Theorem (PSW) As O ranges over all perfect orientations of G, the cone G_O are the max cones in a complete fan F_G.

In fact, F_G is normal to a certain polytope P_G.

Let X_G be the toric variety associated to P_G. We'll give a rational map $X_G \rightarrow (Gr_{kn})_{\geq 0}$ which is well-defined on $(X_G)_{\geq 0}$

Real positive Non-neg. parts of a toric variety

Def Let $Q \subset \mathbb{R}^n$ be a lattice polytope & let $\{\underline{m}_i\}_{i=1}^{\ell}$ be the lattice points inside $Q \cap \mathbb{Z}^n$.

Consider map $\phi : (\mathbb{C}^*)^n \longrightarrow \mathbb{P}^{\ell-1}$ s.t.

$$(x_1, \dots, x_n) \longmapsto [\underline{x}^{\underline{m}_1} : \cdots : \underline{x}^{\underline{m}_\ell}]$$

The toric variety $X_Q = \mathrm{im}\,\phi$.

The real part $X_{Q(\mathbb{R})} = X_Q \cap \mathbb{RP}^{\ell-1}$

The positive part $(X_Q)_{>0}$ is image of $(\mathbb{R}_{>0})^n$ under ϕ

The non-neg part $(X_Q)_{\geq 0}$ is the closure of $(X_Q)_{>0}$ in $X_{Q(\mathbb{R})}$

Fact: $(X_Q)_{\geq 0}$ is homomorphic to Q via moment maps.

Thm (PSW) In fact the parametrization of the cell C_G extends to give a map from

The non-neg part of toric variety $\longrightarrow \overline{C_G} \subset (Gr_{kn})_{\geq 0}$

$\wr\wr$

closed ball

$\Longrightarrow$ cell decomp is CW complex.