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Multigraded Hilbert Schemes joint with Diane Maclagan

There is a ~~point~~ parameter space for all ideals in
$S = \mathbb{C}[x_1, \dots, x_n]$ with a fixed Hilbert function.

AIM: Understand the geometry of these spaces.

Fix an abelian group A. An A-grading of S is given by group homomorphisms $\deg : \mathbb{Z}^n \longrightarrow A$;

$$S = \bigoplus_{a \in A} S_a$$ where S_a is the span of all polynomials of degree a.

A homogeneous ideal I is admissible if $\dim_{\mathbb{C}} (S/I)_a < \infty$ for all $a \in A$; its Hilbert function $h : A \longrightarrow \mathbb{N}$ is

$$h_{S/I}(a) = \dim_{\mathbb{C}} (S/I)_a .$$

HAIMAN–STURMFELS: There exists a quasiprojective scheme Hilb^h parametrizing all ideals in S with Hilbert function h.

EXAMPLE: Let $n=2$

- If $A = 0$, then Hilb^h parametrizes $h(0)$ points in $\mathbb{A}^2$.
- If $A = \mathbb{Z}$, $\deg(x_1) = \deg(x_2) = 1$ and $h(a) = 1$ for $a \geq 0$ then $\mathrm{Hilb}^h \cong \mathbb{P}^1$.
- If $A = \mathbb{Z}$, $\deg(x_1) = -\deg(x_2) = 1$ and $h(a) = 1$ for all $a \in A$, then $\mathrm{Hilb}^h \cong \mathbb{A}^1$.
- If $A = \mathbb{Z}^2$, $\deg(x_1) = (1,0)$ $\deg(x_2) = (0,1)$, then Hilb^h is either empty or a point.

PROPOSITION: Let $q \in \mathbb{Q}[z]$. If $A = \mathbb{Z}$, $\deg(x_j) = 1$ for $1 \leq j \leq n$ and
$$h(a) = \begin{cases} h_S(a) & a < a_0 \\ q(a) & a \geq a_0 \end{cases} \quad a_0 \gg 0 \text{ fixed},$$
then Hilb^h parametrizes subschemes of $\mathbb{P}^{n-1}$ with Hilbert polynomial q. Hence, classic Hilbert Scheme is a multigraded Hilbert scheme.

BAD NEWS

IARROBINO: For $n > 2$ and $A = 0$, Hilb^h is reducible for $h(0) \gg 0$.

VAKIL: Every singularity type appears in the classic Hilbert Scheme.

SANTOS: There exists a disconnected Hilb^h where h is the incidence function of $\deg(\mathbb{N}^n)$.

Such Hilb^h are called toric Hilbert schemes.

GOOD NEWS

HARTSHORNE: The classic Hilbert Scheme is connected.

FOGARTY: If $n=2$ and $A=0$, then Hilb^h is smooth and irreducible.

MACLAGAN-THOMAS: If $\mathrm{rank}(A) \geq n-2$, then the toric Hilbert Scheme is also smooth and irreducible.

HAIMAN-STURMFELS CONJECTURE: If $n=2$, then Hilb^h is smooth and irreducible.

EVAIN: If $A = \mathbb{Z}$, $\deg(x_1)$, $\deg(x_2) \in \mathbb{N}$, then Hilb^h is an irreducible component of a $\mathbb{C}^*$-fixed point set of the classical Hilbert scheme of points.

GOAL: Prove this conjecture.

FACTORING

PROPOSITION: If $H_1, \dots, H_e$ are connected components of Hilb^h then there exists $d \in A$ and $h : A \to \mathbb{N}$ such that

$$H_j \cong \mathbb{P}^{h(d)-1} \times H_j'$$

where H_j' is a connected component of $\mathrm{Hilb}^{h'}$ and $\mathrm{Hilb}^{h'}$ parametrizes ideal with codimension greater than 1.

For $n=2$, reduce the conjecture to the study of finite colength ideals.

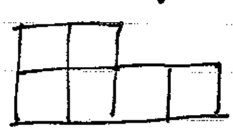
SMOOTHNESS

IVERSON: If G is a smooth, linear, semisimple group acting on a smooth, separated scheme X, then X^G is smooth.

For $n=2$, we can also prove smoothness using Hilbert–Burch theorem and infinitesimal lifting.

For the conjecture, it suffices to prove connectedness.

Gröbner theory implies that it is enough to connect monomial ideals.

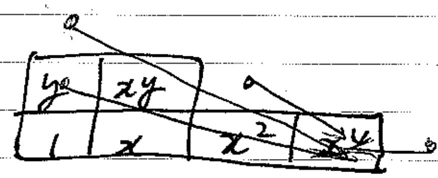
 $\longleftrightarrow I$

TANGENT DIRECTIONS

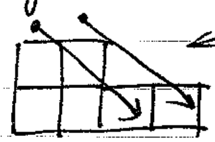
HAIMAN gives a combinatorial model for a basis of tangent space at a point $I \in \mathrm{Hilb}^h$. Bases elements corresponds to arrows with tail on "the border" of I and head in the partition of I.

EXAMPLE : $x_1 = x$, $x_2 = y$

$$I = \langle x^4, x^2y, y^2 \rangle$$



Horizontal and vertical shift of an arrow don't change the basis element provided the head stays in the partition and tail stays on the border.



← equivalent arrows

The collection of arrows is $T(I)$.
We partition $T(I)$ into $T^+(I)$ and $T^-(I)$.

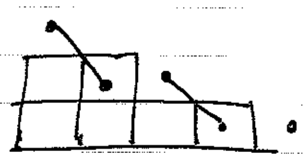
$T^+(I)$ consists of arrows (r,s) to (u,v) such that

$$x^r y^s <_{lex} x^u y^v.$$

CONNECTEDNESS

PROPOSITION: If I is a monomial ideal with $T^+(I) \neq \phi$, then there exists an ideal J with exactly two initial ideals I and I'.

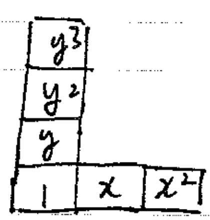
EXAMPLE



$$I = \langle x^4, x^2y, y^2 \rangle$$

$$J = \langle x^4, x^2y - x^3, y^2 - xy \rangle$$

$$I' = \langle x^3, xy, y^4 \rangle$$



PROPOSITION : there exists a unique monomial ideal I with $T^+(I) = \phi$.

THEOREM : For $n=2$, Hilb^h is smooth and irreducible.