Equivalence of Mirror Families Constructed from Toric Degenerations of Flag Varieties

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Definition. *Let* $G = SL_{n+1}(\mathbb{C})$ *and* B *the subgroup of upper triangular matrices. Then we say say that* G/B *is the complete flag variety of type* A_n*.* $(n \geq 2)$

The Weyl group of G is the symmetric group S_{n+1} with simple reflections denoted $s_1, \cdots, s_n$. Let ω_0 denote the unique element of S_{n+1} of maximal length.

There are many different ways of writing ω_0 as a product of simple reflections.

Definition. *We call a* $\overline{\omega_0} = s_{i_1} s_{i_2} \cdots s_{i_N}$ *a reduced decomposition of* ω_0 *if* $\overline{\omega_0} = \omega_0$ *and* $N = \dfrac{n(n+1)}{2}$.

Lemma. *Generic elements of* $|-K_{G/B}|$ *are smooth Calabi-Yau varieties.*

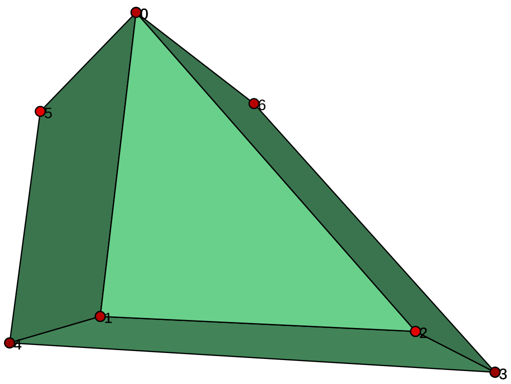
Question. *Can we find a family of varieties mirror to these generic anti-canonical hypersurfaces?*

Answer. **YES**! *By constructions of Givental, Batyrev (et.al.). We review the Batyrev's construction which uses* small toric degenerations *of* G/B.

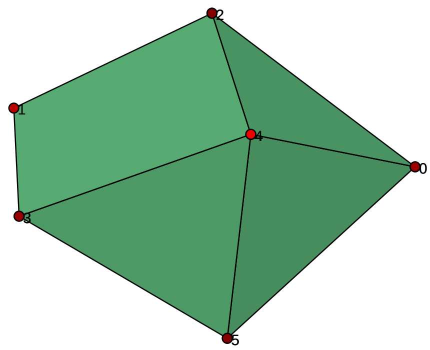
Definition. *A normal Gorenstein toric Fano variety* $Y \subset \mathbb{P}^m$ *is called a* small toric degeneration *of* X, *if there exists a Zariski open neighborhood* U *of* $0 \subset \mathbb{A}^1$ *and an irreducible subvariety* $\widetilde{X} \subset \mathbb{P}^m \times U$ *such that the morphism* $\pi : \widetilde{X} \mapsto U$ *is flat and:*

1. *the fiber* $X_t := \pi^{-1}(t) \subset \mathbb{P}^m$ *is smooth for all* $t \in U \backslash 0$;
2. *the special fiber* $X_0 := \pi^{-1}(0) \subset \mathbb{P}^m$ *has at worst Gorenstein terminal singularities and* X_0 *is isomorphic to* $Y \subset \mathbb{P}^m$;
3. $\mathrm{Pic}(\widetilde{X}/U) \cong \mathrm{Pic}(X_t)$ *for all* $t \in U$.

Given a *small toric degeneration* of the pair $(G/B, -K_{G/B})$ to a toric variety X_Δ corresponding to a reflexive polytope Δ. We can take look at the toric variety associated to the dual polytope Δ^* in $M_\mathbb{R}$.



Δ



Δ^*

We can view M as the lattice of monomials in $\mathbb{C}[t_1, t_1^{-1}, \cdots, t_N, t_N^{-1}]$.

Definition. *$V(X_{\Delta^*})$ is the family of hypersurfaces in $T = Spec(\mathbb{C}[t_1, t_1^{-1}, \cdots, t_N, t_N^{-1}])$ satisfying the equations* $1 = \sum_{\text{vertices } m_i \text{ of } \Delta^*} a_i T^{m_i}$ *where $a_i \in \mathbb{C}^*$.*

Conjecture. *(Batyrev) Generic elements of the subfamily of $V(\Delta^*)$ whose coefficients satisfy a set of relations called* box equations *are birational to mirrors of generic elements of $|-K_{G/B}|$.*

Construction. *(Caldero-Alexeev-Brion) For any choice of reduced decomposition* $\overline{\omega_0}$ *there exists a degeneration of the pair* $(G/B, -K_{G/B})$ *to a toric pair* $(X_\Delta, \mathcal{O}_{X_\Delta}(1))$ *corresponding to a polytope* $\Delta = \Delta(\overline{\omega_0})$*. We call these degenerations* string degenerations*.*

Question. *What is known about these degenerations and their corresponding polytopes?*

1. The polytope Δ is conjectured to be integral.

2. * The dual polytope Δ^* is integral.

3. The f-vectors for different Δ vary greatly, but the number of integral points remains the same.

4. For $\overline{\omega_0} = s_1 s_2 s_1, \cdots, s_n s_{n-1} \cdots s_1$ corresponds to the sagbi, or Gonciulea/Lakshmibai, degeneration which was used in the original mirror construction of Batyrev (et. al.).

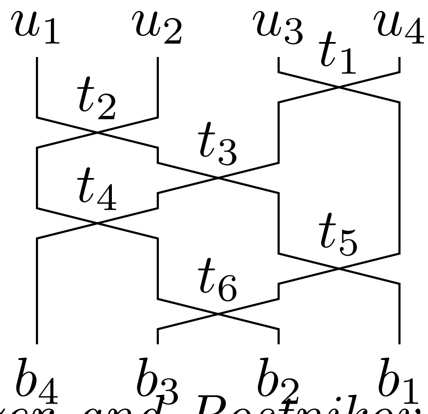
5. * There exist examples of string degenerations which aren't small.

Question. *Can we construct mirror families using the string degenerations?*
Question. *How do these mirror families depend on the choice of* $\overline{\omega_0}$*?*

We attempt to mimic Batyrev's construction
Definition. *Since* Δ^* *is an integral polytope we can still define* $V(X_{\Delta^*})$
Question. *What are the appropriate box equations to use in the string degeneration?*

Note that the box equations correspond to relationships between the facets of Δ. So we need to understand these facets.
Fact. Δ *is the intersection of two polyhedral cones known respectively as the string and* λ*-cones. The sting cone was defined by Berenstein and Zelevinsky, and given a combinatorial description by Gleizer and Postnikov. The* λ*-cone was defined by Littelmann.*

Fix $\overline{\omega_0}$ and draw a string diagram: For example the string diagram for $n = 3$ and reduced word decomposition $\overline{\omega_0} = s_3 s_1 s_2 s_1 s_3 s_2$:

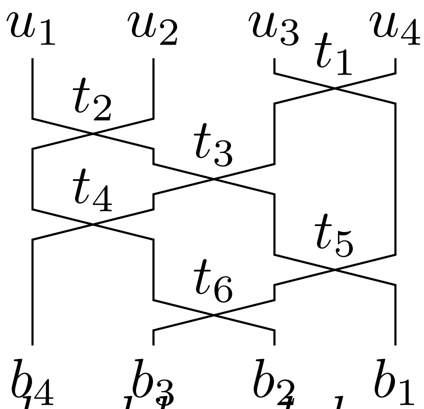


Theorem. *(Gleizer and Postnikov) The string cone can be completely described in terms of certain types of oriented paths on this graph known as* rigorous paths*. Each rigorous path defines a facet of the string cone.*

Lemma. *The λ-cone can be completely described in terms of the combinatorics of this graph. Every point of intersection t_i determines a facets of the λ-cone.*

Definition. *Given a rigorous path p or an intersection point t_i we denote the corresponding monomial in the equations defining $V(X_{\Delta^*})$ by T^p and T^{λ_i}.*

Combinatorial Box equations:



Definition. *The closed bounded regions of the string cone are called boxes. Each box has a top vertex and a bottom vertex. We denote the corresponding coordinates* t_{top} *and* t_{bot}.

Definition. *Let* $T^{\lambda_{top}}$ *(resp.* $T^{\lambda_{bot}}$*) be the monomial corresponding to the* λ*-inequality associated to* t_{top} *(resp.* t_{bot}*).*

Definition. *For every two string inequalities* p_1 *and* p_2 *with corresponding monomials* T^{p_1} *and* T^{p_2} *satisfying the following* box conditions*:*

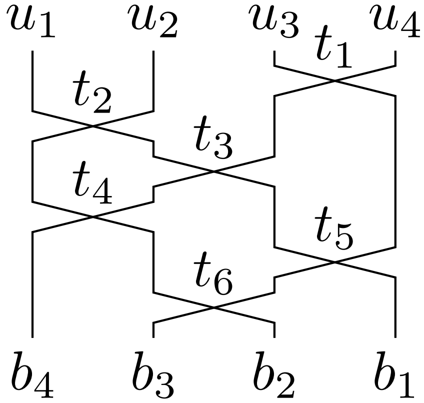
1. *there exists a box with corresponding monomials* $T^{\lambda_{top}}$ *and* $T^{\lambda_{bot}}$ *such that* $T^{p_1}\, T^{\lambda_{top}} = T^{p_2}\, T^{\lambda_{bot}}$*,*

2. *the* t_{top} *degree of* $T^{p_1} = -1$*,*

3. *the* t_{bot} *degree of* $T^{p_2} = 1$*,*

we define an equation $a_{p_1} a_{\lambda_{top}} = a_{p_2} a_{\lambda_{bot}}$*. We call the collection of all such equations the* combinatorial box equations.

$\overline{\omega_0} = s_3 s_1 s_2 s_1 s_3 s_2$:



Definition. *Let* $F_{\overline{\omega_0}}$ *be the subfamily of hypersurfaces in* $V(X_{\Delta^*})$ *whose coefficients satisfy the combinatorial box equations.*

Theorem. *When the string degenerations is small, this is precisely the family defined by Batyrev.*

Question. *Do the families* $F_{\overline{\omega_0}}$ *make good mirror candidates when the degeneration isn't small? And how do they compare for different choices of* $\overline{\omega_0}$*?*

Every two reduced word decompositions $\overline{\omega_0}$ and $\overline{\omega_0}'$ can be linked by a finite sequence of the following two braid move:

1. 2-move: exchanges $(s_i,\ s_j)$ with $(s_j,\ s_i)$ where $|i-j|>1$

2. 3-move: exchanges (s_i, s_j, s_i) with (s_j, s_i, s_j) where $|i-j|=1$

Lustzig gave was a piecewise linear map between $\Delta(\overline{\omega_0})$ and $\Delta(\overline{\omega_0}')$ which differ by a braid move given by:

1. 2-move: $(x_i, x_j) \rightarrow (x_j, x_i)$

2. 3-move: $(x_i, x_j, x_k) \rightarrow$ (max(x_k,$x_j - x_i$),$x_i + x_k$,min(x_i,$x_j - x_k$)). Note that i, j and k are consecutive integers.

Give picture of A_3 case and how they are linked by braid moves.

Theorem. *For $\overline{\omega_0}$ and $\overline{\omega_0}'$ differing by a 2-move. The families $F_{\overline{\omega_0}}$ and $F_{\overline{\omega_0}'}$ are isomorphic.*

Examining how the facets of Δ change under a 3-move will help us define a birational map between the corresponding $F_{\overline{\omega_0}}$ and $F_{\overline{\omega_0}'}$.

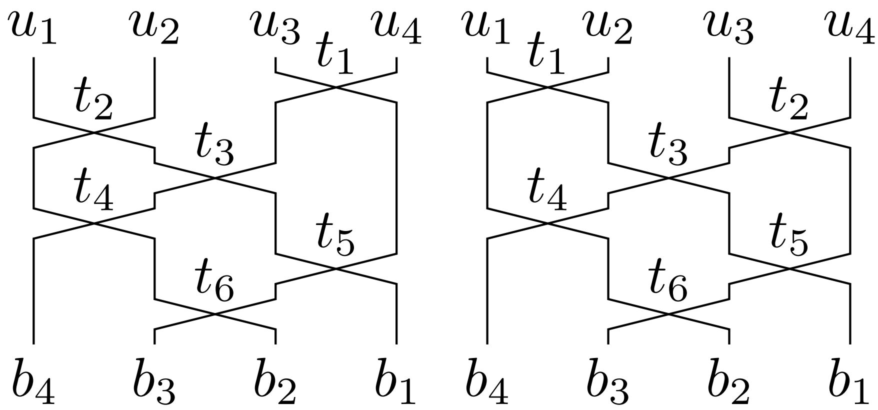


Figure 1: String Diagrams for $\overline{\omega_0} = \mathbf{s_3 s_1} s_2 s_1 s_3 s_2$ and $\overline{\omega_0}' = \mathbf{s_1 s_3} s_2 s_1 s_3 s_2$

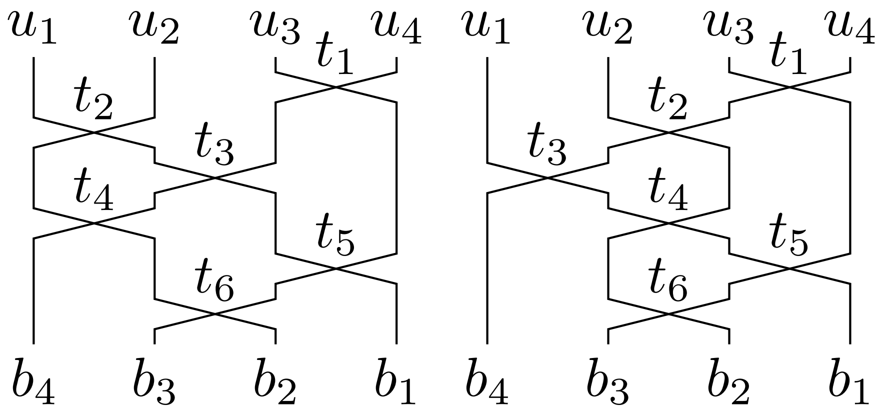
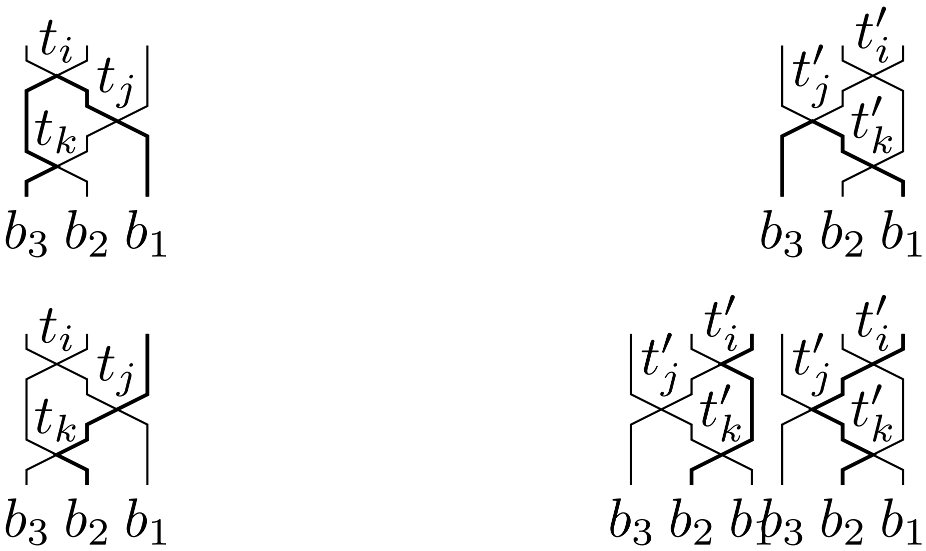


Figure 2: String diagrams for $\overline{\omega_0} = s_3 \mathbf{s_1 s_2 s_1} s_3 s_2$ and $\overline{\omega_0}' = s_3 \mathbf{s_2 s_1 s_2} s_3 s_2$

Theorem. *We have a complete classification of how the* λ *and string inequalities change under a 3-move.*



This understanding of how the inequalities change shows us what $F_{\overline{\omega_0}}$ looks like before and after a braid move.

Theorem. *For any two reduced words* $\overline{\omega_0}$ *and* $\overline{\omega_0}$ *there exists an explicit birational map between* $F_{\overline{\omega_0}}$ *and* $F_{\overline{\omega_0}'}$.

The method of the proof went has follows:

1. Attempt to define a change of coordinates which gives a birational map between $V(X_{\Delta^*})$ and $V(X_{\Delta'^*})$

2. This isn't possible unless we we restrict to a subfamily satisfying box equations whenever Box Condition 1 is met.

3. Try to show that this map preserves the box equations.

4. In order to show this we need that box conditions 2 and three are satisfied.

Since mirror properties are invariant under birational maps we get the answers to our original two questions.

Answer. *The families* $F_{\overline{\omega_0}}$ *are appropriate mirror candidates constructed from the string degenerations*

Answer. *The mirror family is essentially independent of the choice of reduced decomposition.*

Remark. *In the language of tropical geometry, the maps between individual fibers can be given by a particular geometric lift of the piecewise linear map between the string cones.*