CENTRE
DE RECHERCHES
MATHÉMATIQUES

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Kleiman-Bertini theorem for sheaf tensor products

k-field char $p > 0$ allowed, $k \neq \bar{k}$ allowed

G group $\underbrace{\text{variety}}_{\text{reduced}} / k$ $\Rightarrow$ smooth $\qquad G = GL_n$, B, elliptic curve, ...

X G-variety $\quad G \times X \longrightarrow X$

<u>Def</u> G acts transitively on X if $G \times X \twoheadrightarrow X \times X$.

$$(g, x) \longmapsto (gx, x)$$

$G(\bar{k}) \times \{x\} \twoheadrightarrow X(\bar{k}) \cdot \{x\} \quad \forall x \in X(\bar{k})$ in geometric notation

motivating examples: $G = GL_n$

$$X = GL_n / P$$

$g \in G(k) \rightsquigarrow$ automorphism $g : X \to X$.

$\mathcal{F}$ sheaf on $X \longrightarrow g^* \mathcal{F}$.

$\mathcal{E}$ quasicoherent$/X$

$0 \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \cdots$ locally free resolution of $\mathcal{E}$

$\mathcal{L}_i$ locally free

$H_i(\mathcal{L}_\cdot) = 0$ if $i \neq 0$

$H_0(\mathcal{L}_\cdot) = \mathcal{E}$

$\mathcal{E}\,\mathcal{F}$

$H_i(0 \leftarrow \mathcal{F} \otimes \mathcal{L}_0 \leftarrow \mathcal{F} \otimes \mathcal{L}_1 \leftarrow \cdots)$

$=: \mathrm{Tor}_i^X(\mathcal{E}, \mathcal{F})$.

<u>Thm</u> [M-Speyer '05]

Fix X transitive G-variety

$\mathcal{E}, \mathcal{F}$ coherent sheaves$/X$

Then $\exists$ Zariski open dense $U \subseteq G$ such that
$\forall g \in U(k)$

$Tor_i^X(g^*\mathcal{F}) = 0, \ \forall i \geq 1.$

Note: Assume k infinite and G connected, affine
Then $\exists \ g \in U(k)$
if either (i) G is reductive
or (ii) k is perfect.

Kleiman transversality thm
Kleiman – Bertini

Fix X transitive G-variety over k, char $k = 0$.
$Y, Z \subseteq X$ smooth subvarieties.

$\exists$ open dense $U \subseteq G$ such that $\forall g \in U(k)$

Y intersects gZ transverally.

$(\Rightarrow Y \cap gZ$ is smooth$)$

Cor Schubert caculas is positive.

~~pf:~~ To show $[X_w] \cdot \underbrace{[X_v]}_{[X^v]} \cdot [X_u] > 0$

$X_w \cap X^v$ Richardson variety
transverse

$BB_- \subseteq G$ open & dense

Pick $b b_-$ with

$$X_w \pitchfork b b_- X^v = X_w \pitchfork b X^v$$
$$\Leftrightarrow b^+ X_w \pitchfork X^v$$
$$= X_w \pitchfork X^v.$$

(transverse)

$$[X_w][X_v][X_u] = [X_w \cap X^v] \cdot [X_u]$$
$$= [X_w \cap X^v] \cdot [g X_u] \qquad \text{generic } g$$
$$= [X_w \cap X^v \cap g X_u]$$
$$= [\text{finite set of pts}] \quad \text{if } \sum \text{codim} = \dim G/P$$
$$\geq 0$$

Moral for Schubert calculus : $[Y] \cdot [Z] = [Y \cap gZ]$ generic g

$\Rightarrow \mathrm{Tor}_i(\mathcal{O}_Y, \mathcal{O}_{gZ}) = 0$ for $i \geq 1$

Thm $\Rightarrow$ K-theoretic version of this moral.

K-theory

X scheme

$K_0(X) = \mathbb{Z}\{[\mathcal{F}] \mid \mathcal{F} \text{ coherent}/X\} \big/ \langle [\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}''] \text{ if } 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \text{ s.e.s} \rangle$

$(G_0(X))$

$K^0(X)$ and locally free

$K^0(X) \longrightarrow K_0(X)$ "poincaré duality"

Prop $K^0(X) \longrightarrow K_0(X)$ is iso. if X is smooth.

resenlottom

Pf. $0 \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \cdots \leftarrow \mathcal{L}_{\dim(X)} \leftarrow 0$
locally free res of $\mathcal{F}$ $\Rightarrow [\mathcal{F}] = \sum_{i=1}^{\dim(X)} (-1)^i [\mathcal{L}_i]$ $\square$

General X

$K^0(X)$ ring : $[\mathcal{L}][\mathcal{L}'] = [\mathcal{L} \otimes \mathcal{L}']$

$K_0(X)$ module over $K^0(X)$: $[\mathcal{L}][\mathcal{F}] = [\mathcal{L} \otimes \mathcal{F}]$

X smooth $\Rightarrow K^0 \cong K_0$. What's $[\mathcal{E}][\mathcal{F}]$?

$\mathcal{L}_\bullet$ res. of $\mathcal{E} \Rightarrow [\mathcal{E}][\mathcal{F}] = \sum_i (-1)^i [\mathcal{L}_i][\mathcal{F}]$

$$= \sum_i (-1)^i [\mathcal{L}_i \otimes \mathcal{F}]$$

$$= \sum_i (-1)^i [\mathrm{Tor}_i(\mathcal{E}, \mathcal{F})]$$

Thm says $[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E}][g_* \mathcal{F}]$ in G/P

$$= [\mathrm{Tor}_0(\mathcal{E}, g_* \mathcal{F})]$$

$$= [\mathcal{E} \otimes g_* \mathcal{F}] \quad \text{generic } g$$

Generalization [Susan Sierra, 07]

[Speiser 1986] X smooth/k, char $k = 0$, G acts on X

$Z \subseteq X$ smooth. Assume $Z \pitchfork H$ for all G-orbits $H \subseteq X$
Then for any $Y \subseteq X$

$U \subseteq G$ open dense. $\quad Y \pitchfork gZ \quad \forall g \in U(k)$

<u>Def</u> $\mathcal{E}, \mathcal{F}$ coherent $/X$. $\mathcal{F} \pitchfork \mathcal{E}$ <u>homologically transverse</u> if

$$\mathrm{Tor}_i(\mathcal{F}, \mathcal{E}) = 0, \ \forall i \geq 1.$$

<u>Thm</u> [Sierra math AG/0705.0055] k arbitrary
X G-variety. $\mathcal{F}$ coherent $/X$ TFAE

(i) $\mathcal{F} \pitchfork \mathcal{O}_{\bar{H}}$, $\forall$ G-orbits $H \subseteq X$. (trivial)
(ii) $\forall$ coherent $\mathcal{E}/X$ $\exists U \subseteq G$ open dense with

$$\mathcal{E} \pitchfork g_* \mathcal{F} \ \forall g \in U(k).$$

<u>Prop</u> [Sierra]
$$\begin{tikzcd} \mathcal{G} \ W \arrow[r] \arrow[d] & X \\ B & \end{tikzcd}$$
$\mathcal{G}$ coherent $/W$ flat $/X$
$\mathcal{E}$ coherent $/X$
W_b = fiber over $b \in B$
$\mathcal{G}_b = \mathcal{G} \otimes_{\mathcal{O}_W} \mathcal{O}_{W_b}$

Then $\exists$ dense open $U \subseteq B$ such that $\mathcal{E} \pitchfork \mathcal{G}_b$, $\forall b \in U$.

<u>Pf</u> Generic flatness of $\begin{tikzcd} W \arrow[d] \\ B \end{tikzcd}$

$$\mathcal{G} = \mathcal{O}_G \boxtimes \mathcal{F}$$

<u>pf A Thm</u> Apply to
$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & X \\ G & \end{tikzcd} \qquad \begin{tikzcd} (g, x) \arrow[r, mapsto] \arrow[d, mapsto] & gx \\ g & \end{tikzcd}$$

after showing $\mathcal{G}$ flat$/X \Leftrightarrow \mathcal{F} \pitchfork \mathcal{O}_{\bar{H}}$ $\forall$ orbit H

$$\mathcal{E} \pitchfork \mathcal{G}_g \Leftrightarrow \mathcal{E} \pitchfork g_* \mathcal{F} \quad \square.$$