A geometric approach to Carlitz–Dedekind sums

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joint work with

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A little background

The Dedekind eta function under $SL_n(\mathbb{Z})$:

$$\eta(\tau) = \mathbf{e}^{\frac{\pi \imath \tau}{12}} \Pi_{n=1}^{\infty} \left(1 - \mathbf{e}^{2\pi \imath n \tau}\right)$$

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analysis, number theory, combinatorics

q-series
Weierstrass elliptic functions
modular forms
Kronecker limit formula

Richard Dedekind circa 1880

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Definition (Dedekind sum): For relatively prime positive integers a and b,

$$\mathrm{s}\,(a,b) = \sum_{k=1}^{b-1} \left(\left(\frac{ka}{b} \right) \right) \left(\left(\frac{k}{b} \right) \right)$$

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where

$$((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}\ , \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

where do these sums show up?

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analysis, number theory, combinatorics

theta functions

group actions on manifolds

integer-point enumeration in polytopes

back to Dedekind sums

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can take a long time to compute: $\mathrm{s}(3,100)=\sum_{k=1}^{99}\left(\left(\frac{3k}{100}\right)\right)\left(\left(\frac{k}{100}\right)\right)$

Theorem (Dedekind reciprocity):

$$\mathrm{s}\,(a,b)+\mathrm{s}\,(b,a)=-\frac{1}{4}+\frac{1}{12}\left(\frac{a}{b}+\frac{1}{ab}+\frac{b}{a}\right)$$

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Example: $\mathrm{s}(3,100)+\mathrm{s}(100,3)=-\frac{1}{4}+\frac{1}{12}\left(\frac{3}{100}+\frac{1}{300}+\frac{100}{3}\right)$

so, $\sum_{k=1}^{99}\left(\left(\frac{3k}{100}\right)\right)\left(\left(\frac{k}{100}\right)\right)+\sum_{k=1}^{2}\left(\left(\frac{100k}{3}\right)\right)\left(\left(\frac{k}{3}\right)\right)=-\frac{1}{4}+\frac{1}{12}\left(\frac{3}{100}+\frac{1}{300}+\frac{100}{3}\right)$

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Note: $\mathrm{s}(a,b)=\mathrm{s}(a \bmod b,b)$

(Dedekind reciprocity was proved algebraically)

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Definition (Carlitz polynomial): For indeterminates u and v, and relatively prime positive integers a and b,

$$\mathrm{c}\left(u, v; a, b\right) = \sum_{k=1}^{a-1} u^{k-1} v^{\left\lfloor \frac{kb}{a} \right\rfloor}$$

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$\lfloor x \rfloor$ is the greatest integer less than or equal to x ,

$\lfloor x \rfloor = x - \{x\}$

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$$(u-1)\,\mathrm{c}(u,v;a,b) + (v-1)\,\mathrm{c}(v,u;b,a) = u^{a-1}v^{b-1} - 1$$

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$$\Downarrow$$

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$$(u-1)\sum_{k=1}^{a-1} u^{k-1} v^{\left\lfloor \frac{kb}{a} \right\rfloor} + (v-1)\sum_{k=1}^{b-1} v^{k-1} u^{\left\lfloor \frac{ka}{b} \right\rfloor} = u^{a-1}v^{b-1} - 1$$

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$$\sum_{k=1}^{b-1}\left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right) + \sum_{k=1}^{a-1}\left(\left(\frac{kb}{a}\right)\right)\left(\left(\frac{k}{a}\right)\right) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right)$$

Goals

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- We know that Dedekind sums appear naturally in polyhedral geometry; we show that Carlitz polynomials appear naturally as well.
- We give novel geometric proofs of Carlitz' reciprocity theorem, some of its generalizations, and some new reciprocity theorems.
- We realize the equivalence of Carlitz polynomials and the integer-point transform of a two-dimensional analogue of the Mordell–Pommersheim tetrahedron.
- We give an intrinsic geometric reason why Dedekind sums appear in the lattice point enumerator of the tetrahedron by applying Brion's decomposition theorem to the Mordell–Pommersheim tetrahedron.

Motivation

a Carlitz polynomial: $\mathrm{c}(u, v; a, b)$

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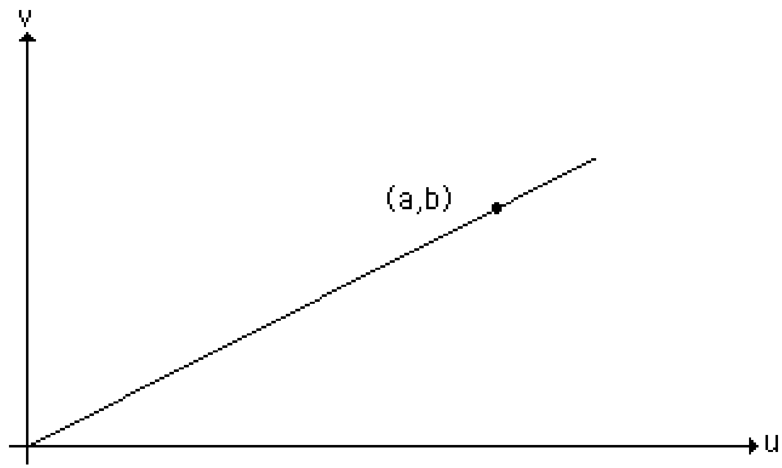

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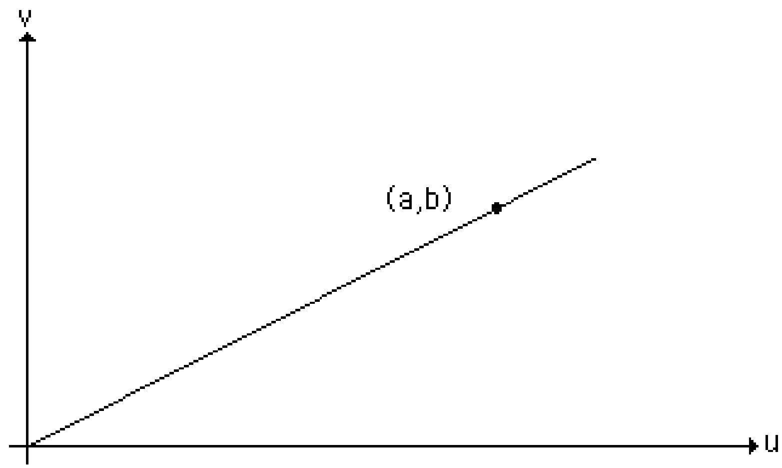

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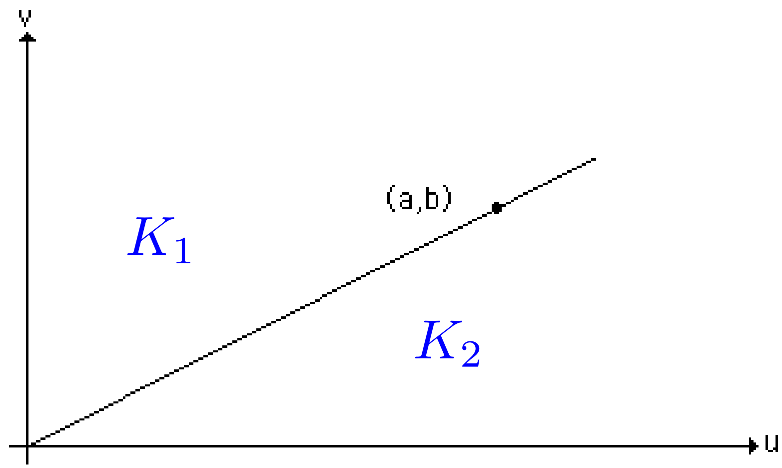
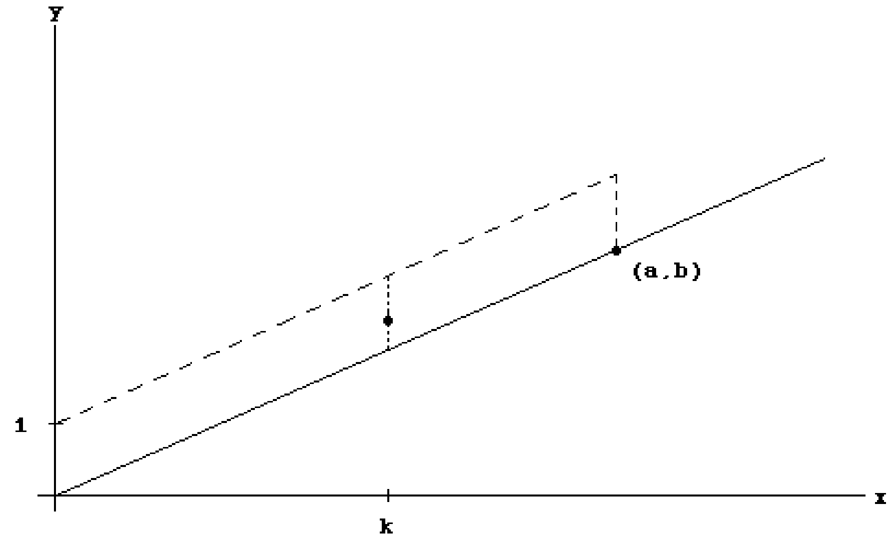


A pointed cone, K , is the intersection of finitely many half-spaces that meet in exactly one point, the vertex.

$$K_1 = \{\lambda_2 \mathbf{e}_2 + \lambda(a, b) : \lambda_2, \lambda \geq 0\} \quad \text{(closed)}$$

$$K_2 = \{\lambda_1 \mathbf{e}_1 + \lambda(a, b) : \lambda_1 > 0, \lambda \geq 0\} \quad \text{(half-open)}$$

How do we list the integer points in each cone?



$$\Pi_1 = \{\lambda_2 \mathbf{e}_2 + \lambda(a,b) : 0 \leq \lambda_2, \lambda < 1\}$$

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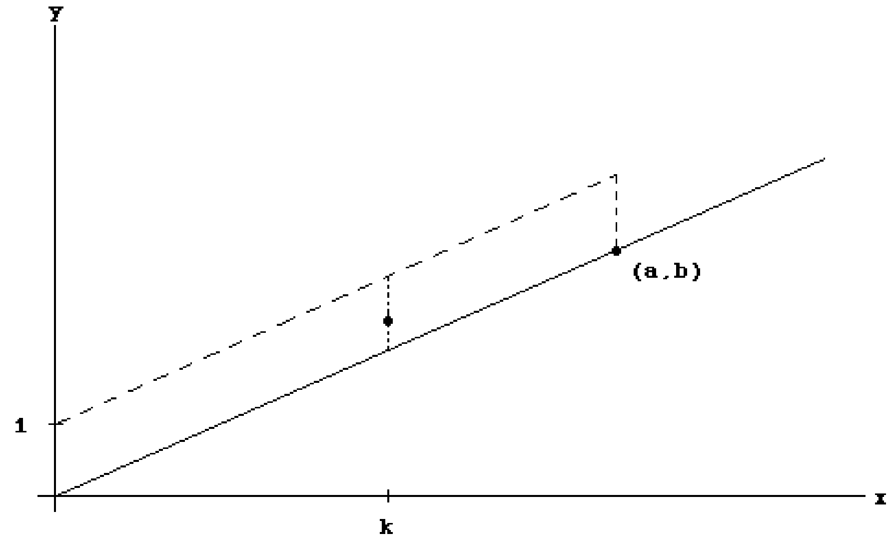

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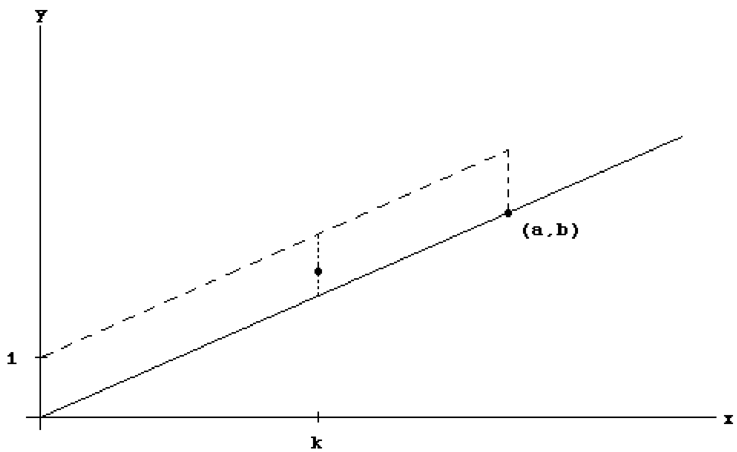
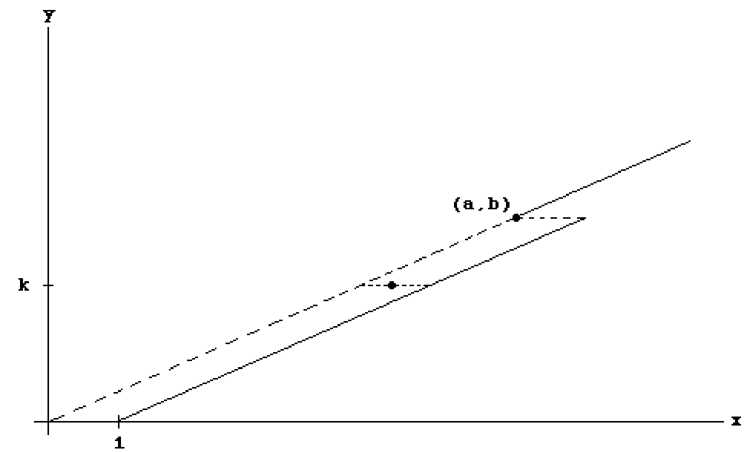
Note:

$$\{(k,y) \in \Pi_1 \cap \mathbb{Z}^2\} = \left\{(0,0), \left(k, \left\lfloor \frac{kb}{a} \right\rfloor + 1\right) : 1 \le k \le a-1\right\}$$

Integer points in the fundamental parallelograms





$$\Pi_1 = \{\lambda_2 \mathbf{e}_2 + \lambda(a,b) : 0 \le \lambda_2, \lambda < 1\}$$
$$\Pi_2 = \{\lambda_1 \mathbf{e}_1 + \lambda(a,b) : 0 < \lambda_1 \le 1, 0 \le \lambda < 1\}$$

and

$$\{(k,y) \in \Pi_1 \cap \mathbb{Z}^2\} = \left\{(0,0), \left(k, \left\lfloor \frac{kb}{a} \right\rfloor + 1\right) : 1 \le k \le a-1\right\}$$
$$\{(x,k) \in \Pi_2 \cap \mathbb{Z}^2\} = \left\{(1,0), \left(\left\lfloor \frac{ka}{b} \right\rfloor + 1, k\right) : 1 \le k \le b-1\right\}$$

Discrete $\longrightarrow$ continuous

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$\nwarrow$ nicer

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Our generating function takes integer points and embeds them as the multidegree of a monomial.

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Definition: If S is a rational polyhedron,

$$\sigma_S(\mathbf{z}) = \sigma_S\left(z_1^{m_1}, z_2^{m_2}, \ldots, z_d^{m_d}\right) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$$

is called the integer-point transform of S .

The integer-point transform of our vertex cones

The integer points inside Π_1 are $\left(k, \left\lfloor \frac{kb}{a} \right\rfloor + 1\right)$ for $1 \leq k \leq a-1$ and are encoded in the generating function as

$$u^0 v^0 + \sum_{k=1}^{a-1} u^k v^{\left\lfloor \frac{kb}{a} \right\rfloor + 1}$$

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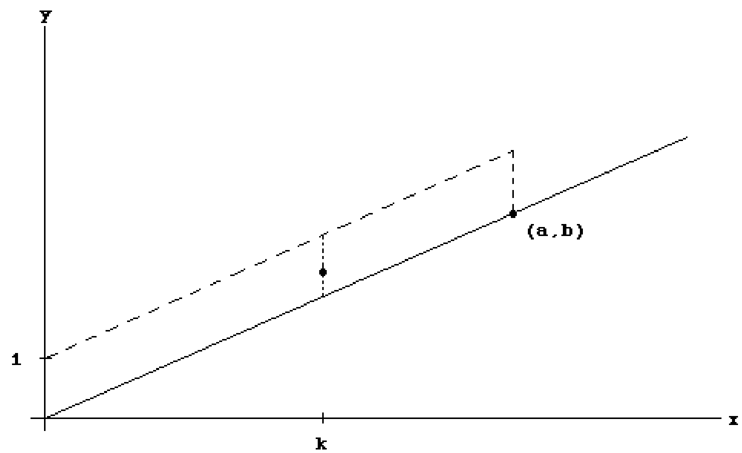
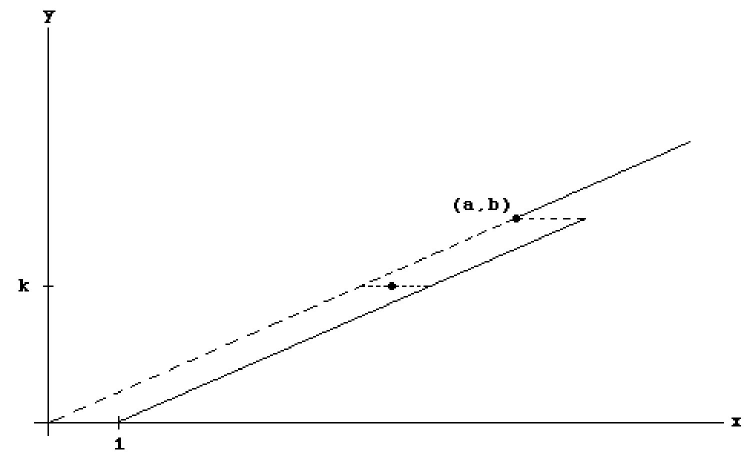
Therefore,

$$\sigma_{K_1} = \frac{1 + uv\, \mathrm{c}(u, v; a, b)}{(1-v)\,(1 - u^a v^b)}$$

So we have





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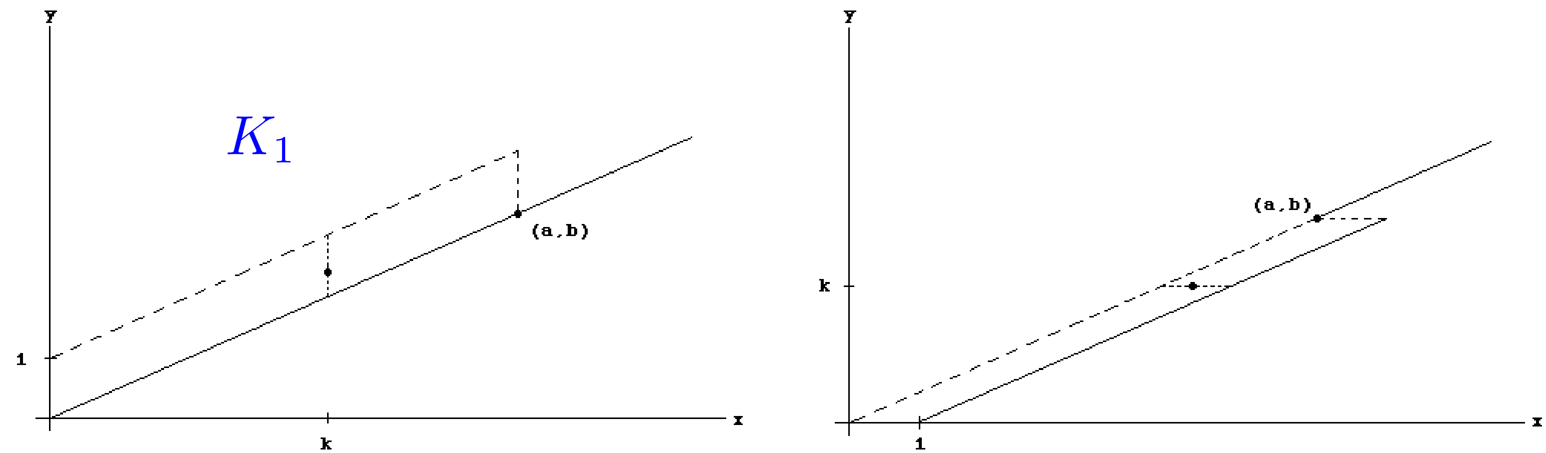

$$\sigma_{K_1}(u,v) = \frac{1 + uv\,\mathrm{c}(u,v;a,b)}{(v-1)(u^a v^b - 1)}$$

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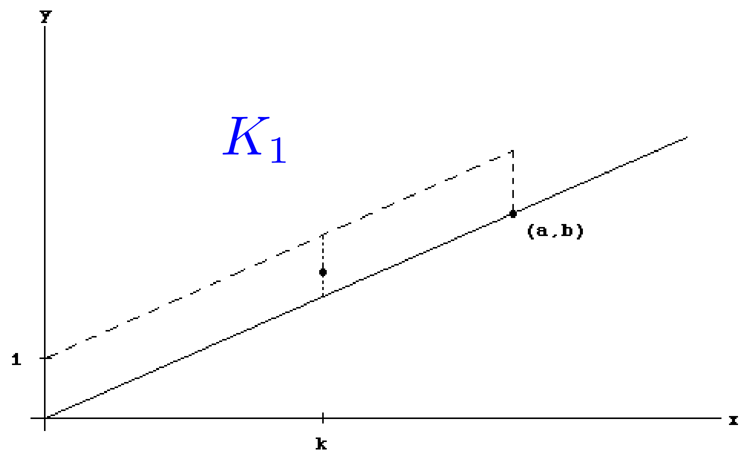

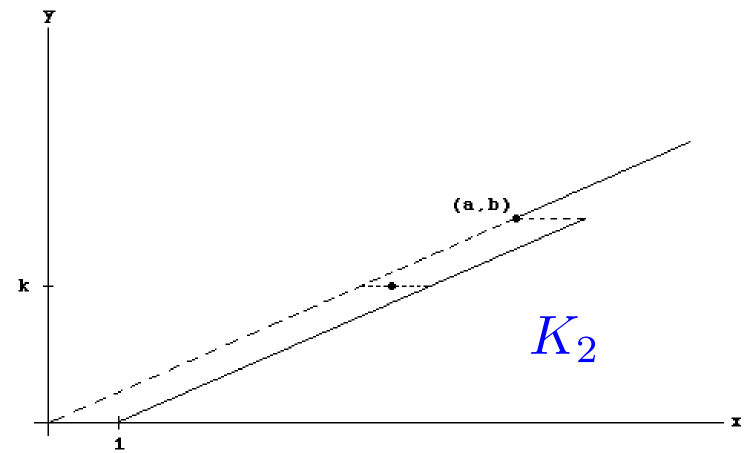

$$\sigma_{K_1}(u,v) = \frac{1 + uv\,\mathrm{c}(u,v;a,b)}{(v-1)(u^a v^b - 1)}$$

and

$$\sigma_{K_2}(u,v) = \frac{u + uv\,\mathrm{c}(v,u;b,a)}{(u-1)(u^a v^b - 1)}$$

Putting it together

$$\sigma_{K_1}(u,v) + \sigma_{K_2}(u,v) = \sigma_Q(u,v)$$

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$$\Rightarrow (u-1)\,\mathrm{c}(u,v;a,b) + (v-1)\,\mathrm{c}(v,u;b,a) = u^{a-1}v^{b-1} - 1$$

Carlitz reciprocity in n dimensions

Definition:

$$\mathrm{c}\left(u_1, u_2, \ldots, u_n; a_1, a_2, \ldots, a_n\right) := \sum_{k=1}^{a_1-1} u_1^{k-1} u_2^{\left\lfloor \frac{ka_2}{a_1} \right\rfloor} u_3^{\left\lfloor \frac{ka_3}{a_1} \right\rfloor} \cdots u_n^{\left\lfloor \frac{ka_n}{a_1} \right\rfloor}$$

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Theorem (Berndt–Dieter): If $a_1, a_2, \ldots, a_n$ are pairwise relatively prime positive integers, then

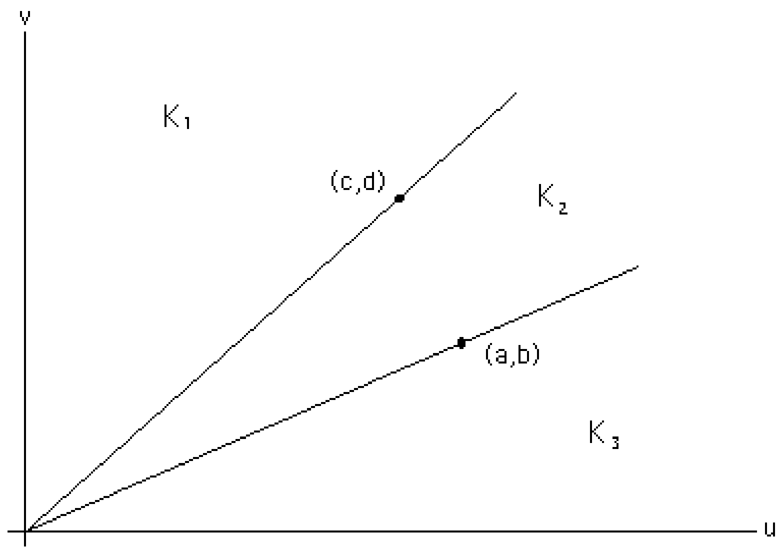
$$\begin{aligned}
&(u_1 - 1)\,\mathrm{c}\left(u_1, u_2, \ldots, u_n; a_1, a_2, \ldots, a_n\right) \\
&\quad + (u_2 - 1)\,\mathrm{c}\left(u_2, u_3, \ldots, u_n, u_1; a_2, a_3, \ldots, a_n, a_1\right) \\
&\quad + \cdots + (u_n - 1)\,\mathrm{c}\left(u_n, u_1, \ldots, u_{n-1}; a_n, a_1, \ldots, a_{n-1}\right) \\
&= u_1^{a_1-1} u_2^{a_2-1} \cdots u_n^{a_n-1} - 1.
\end{aligned}$$

What else can we do?

Other geometric pictures

Other geometric pictures

Two rays in the first quadrant:



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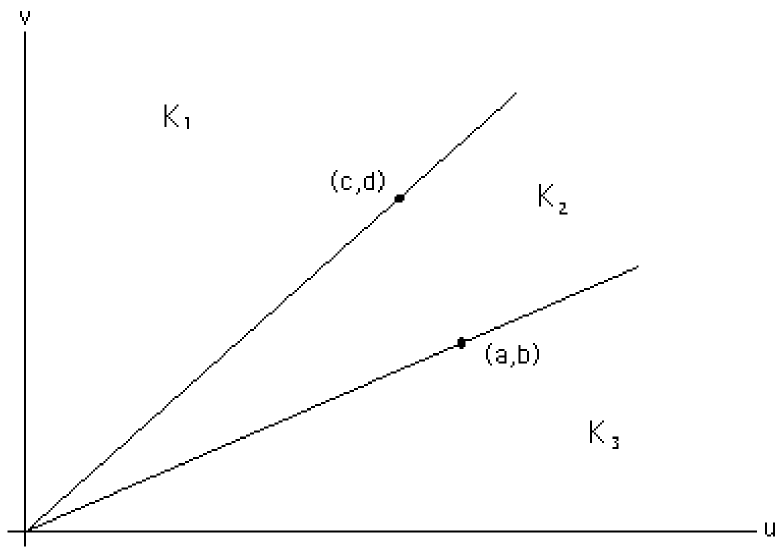

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$$

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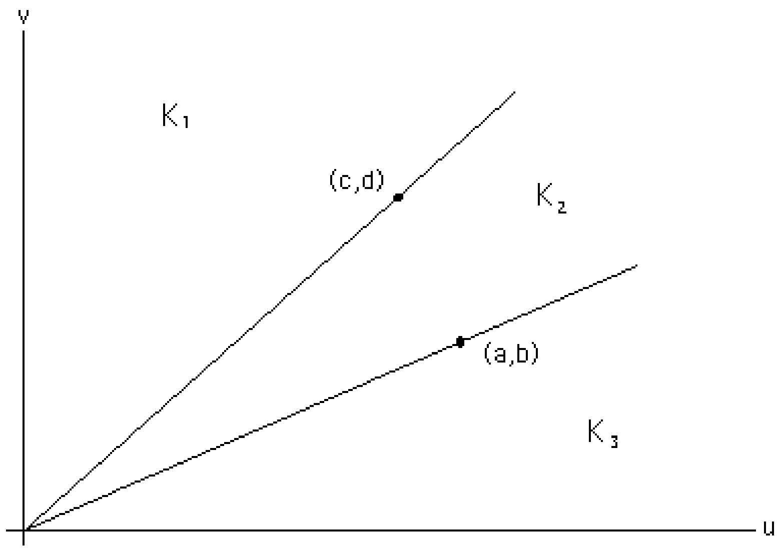

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$$

Theorem (Beck, Matthews):

$$\begin{aligned}(u-1)\left(u^a v^b - 1\right) \mathrm{c}\,(u, v; c, d) + (v-1)\left(u^c v^d - 1\right) \mathrm{c}\,(v, u; b, a) \\ = u^{a+c-1} v^{b+d-1} - u^a v^b - u^c v^d + u^{a-1} v^b + u^c v^{d-1} \\ -u^{a-1} v^{b-1} - u^{c-1} v^{d-1} + 1\end{aligned}$$

Other geometric pictures

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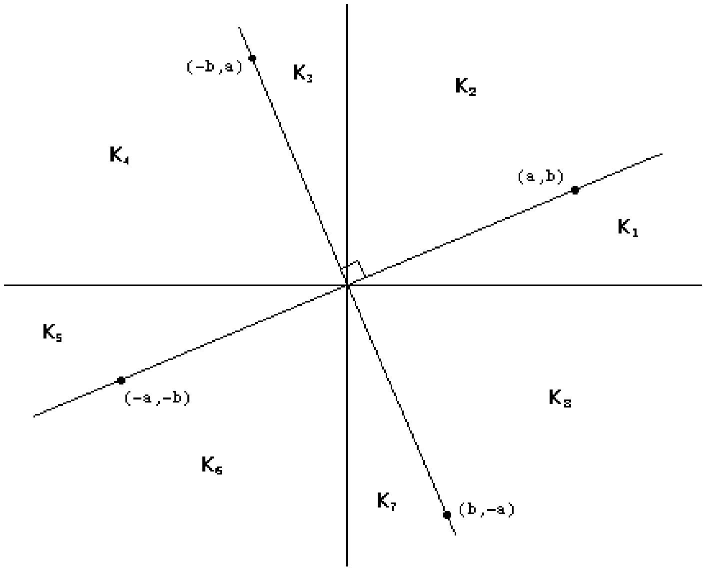
Perpendicular rays in the plane:



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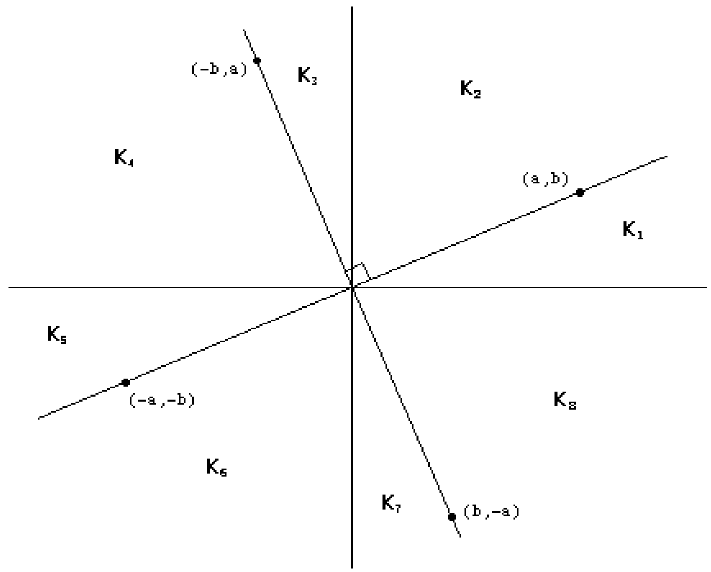


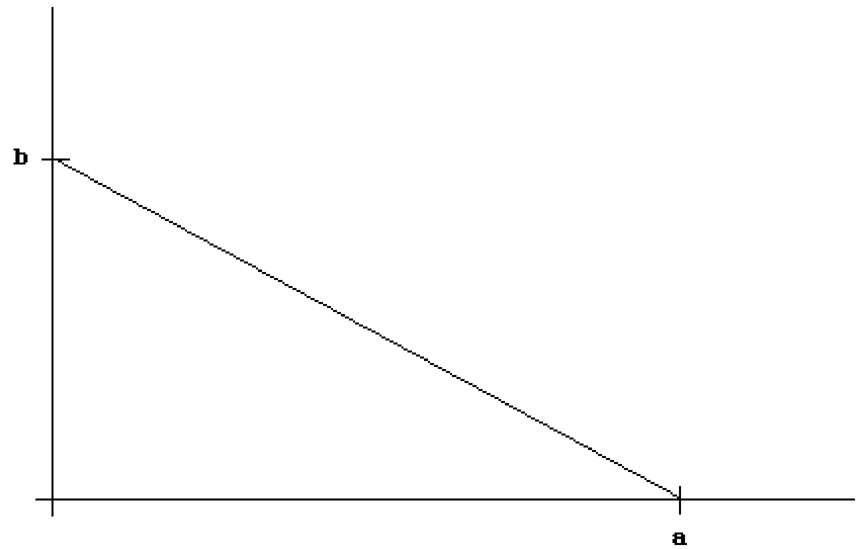
Theorem (Beck,M):

$$\begin{aligned} uv^{-1}(v-1)\left(u^{-b}v^{a}-1\right)\mathrm{c}\left(v^{-1},u;a,b\right) \\ +u^{-1}v(u-1)\left(u^{b}v^{-a}-1\right)\mathrm{c}\left(u^{-1},v;b,a\right) \\ = u\left(u^{-b}v^{a}-1\right)+v\left(u^{b}v^{-a}-1\right) \end{aligned}$$

Other geometric pictures

Other geometric pictures

A triangle:



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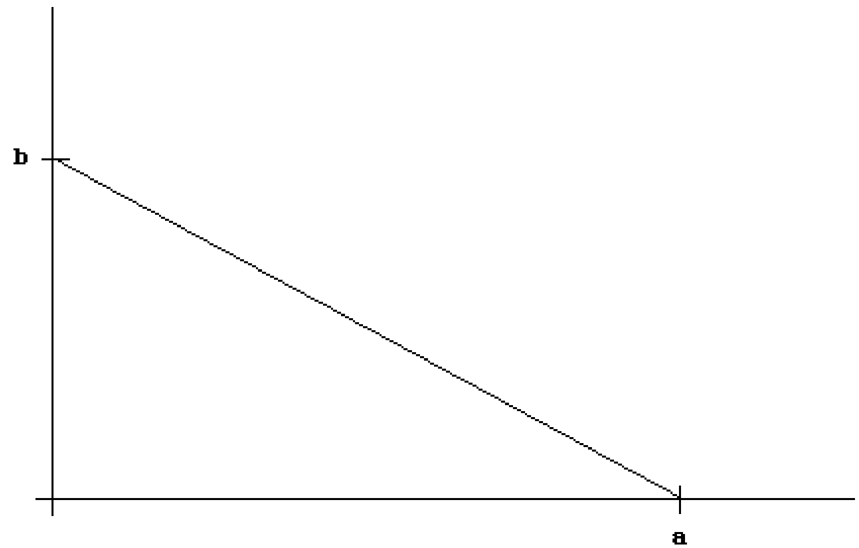


Definition: A vertex cone $K_{\mathbf{v}}$ of a polytope P is the smallest cone with vertex $\mathbf{v}$ that contains P.

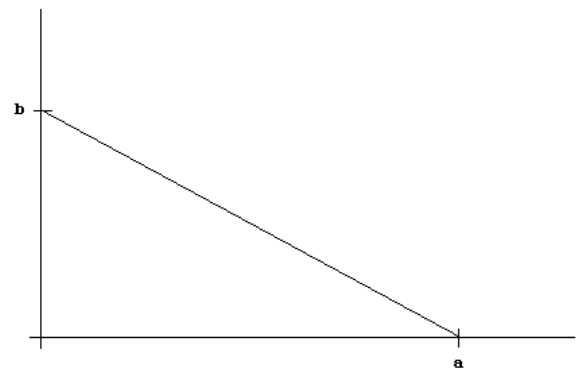
Theorem (Brion 1988): Suppose P is a rational convex polytope. Then we have the following identity of rational functions:

$$\sigma_P(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } P} \sigma_{K_\mathbf{v}}(\mathbf{z})$$

where $\mathbf{z} := (z_1, z_2, \ldots, z_n)$.

The triangle

Consider the triangle P :



The integer-point transform of the vertex cone $K_{(a,0,0)}$ is given by

$$\sigma_{K_{(a,0,0)}}(u,v) = -\frac{u^{a+1} + u^a v\,\mathrm{c}\left(v, u^{-1}; b, a\right)}{(u-1)\left(u^{-a}v^b - 1\right)}$$

Brion's theorem applied to the triangle

Then using Brion's theorem we have

$$\sigma_P(u,v) \quad = \quad \sigma_{K_{(a,0,0)}}(u,v) + \sigma_{K_{(0,b,0)}}(u,v) + \sigma_{K_{(0,0,0)}}(u,v)$$

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$$\begin{aligned}\sigma_P(u,v) &= \sigma_{K_{(a,0,0)}}(u,v)+\sigma_{K_{(0,b,0)}}(u,v)+\sigma_{K_{(0,0,0)}}(u,v)\\ &= -\frac{u^{a+1}+u^a v\,\mathrm{c}\left(v,u^{-1};b,a\right)}{(u-1)\left(u^{-a}v^b-1\right)}-\frac{v^{b+1}+uv^b\,\mathrm{c}\left(u,v^{-1};a,b\right)}{(v-1)\left(u^a v^{-b}-1\right)}\\ &\quad +\frac{1}{(u-1)(v-1)}\end{aligned}$$

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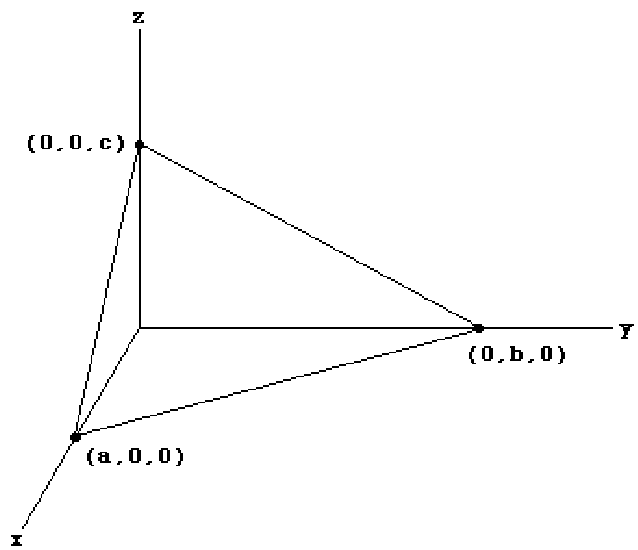
$$\begin{aligned}\sigma_P(u,v) &= \sigma_{K_{(a,0,0)}}(u,v) + \sigma_{K_{(0,b,0)}}(u,v) + \sigma_{K_{(0,0,0)}}(u,v) \\ &= -\frac{u^{a+1} + u^a v\,\mathrm{c}\left(v, u^{-1}; b, a\right)}{(u-1)\left(u^{-a}v^b - 1\right)} - \frac{v^{b+1} + uv^b\,\mathrm{c}\left(u, v^{-1}; a, b\right)}{(v-1)\left(u^a v^{-b} - 1\right)} \\ &\quad + \frac{1}{(u-1)(v-1)}\end{aligned}$$

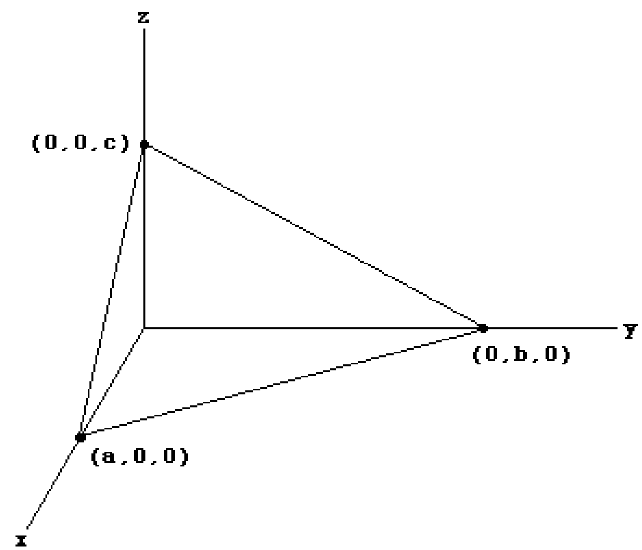
Theorem (Beck, M):

$$(u-1)\sigma_P(u,v) = u^a v\,\mathrm{c}\left(v, u^{-1}; b, a\right) + u\left(u^a + v^b\right) - \frac{v^{b+1} - 1}{v-1}$$

A tetrahedron



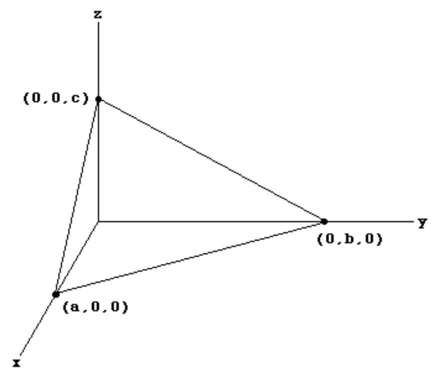
A tetrahedron



Definition (DRC sum):

$$\bar{c}(u,v,w;a,b,c) := \sum_{k=0}^{c-1}\sum_{j=0}^{b-1} u^{\left\lfloor \frac{ja}{b}+\frac{ka}{c}\right\rfloor} v^j w^k$$

A tetrahedron



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Integer-point transform:

$$\sigma_{tK_{(a,0,0)}}(u,v,w) = \frac{u^{(t+2)a}\left[(u-1)+\bar{\mathrm{c}}\left(u^{-1},v,w;a,b,c\right)\right]}{(u-1)\left(u^a-v^b\right)\left(u^a-w^c\right)}$$

Brion's theorem applied to the tetrahedron

Theorem (Beck, M):

$$\begin{aligned}
&(u-1)(v-1)(w-1)\left(u^a - v^b\right)\left(u^a - w^c\right)\left(v^b - w^c\right)\sigma_{tP}(u,v,w) \\
&= u^{(t+2)a}(v-1)(w-1)\left(v^b - w^c\right)\left[(u-1) + \bar{\mathrm{c}}\left(u^{-1}, v, w; a, b, c\right)\right] \\
&\quad - v^{(t+2)b}(u-1)(w-1)\left(u^a - w^c\right)\left[(v-1) + \bar{\mathrm{c}}\left(v^{-1}, u, w; b, a, c\right)\right] \\
&\quad + w^{(t+2)c}(u-1)(w-1)\left(u^a - v^b\right)\left[(w-1) + \bar{\mathrm{c}}\left(w^{-1}, u, v; c, a, b\right)\right] \\
&\qquad - \left(u^a - v^b\right)\left(u^a - w^c\right)\left(v^b - w^c\right)
\end{aligned}$$

Something to think about

Theorem (Mordell–Pommersheim 1951, 1993): Let tP be the dilated tetrahedron and let a, b and c be pairwise relatively prime. Then

$$\begin{aligned} L_{tP}(t) \quad &= \quad \frac{abc}{6}\,t^3 + \frac{ab+ac+bc+1}{4}\,t^2 + \left(-\,\mathrm{s}(bc,a) - \mathrm{s}(ca,b) - \mathrm{s}(ab,c)\right)t \\ &\quad + \left(\frac{3}{4} + \frac{a+b+c}{4} + \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + \frac{1}{abc}\right)\right)t + 1 \end{aligned}$$

where $L_{tP}(t)$ is the lattice-point enumerator for the t^{th} dilate of $P \subset \mathbb{R}^d$ and is equivalent to $\#\left(tP \cap \mathbb{Z}^d\right)$, the discrete volume of tP .

DRC to Dedekind

$\sigma_{tP}(1, 1, 1) = L_{tP}(t)$

DRC to Dedekind

$$\sigma_{tP}(1,1,1) = L_{tP}(t)$$

$$\sigma_{tP}(u,v,w) =$$

$$\begin{gathered}
u^{(t+2)a}(v-1)(w-1)\left(v^b - w^c\right)\left[(u-1) + \bar{\mathrm{c}}\left(u^{-1}, v, w; a, b, c\right)\right] \\
-v^{(t+2)b}(u-1)(w-1)\left(u^a - w^c\right)\left[(v-1) + \bar{\mathrm{c}}\left(v^{-1}, u, w; b, a, c\right)\right] \\
+w^{(t+2)c}(u-1)(w-1)\left(u^a - v^b\right)\left[(w-1) + \bar{\mathrm{c}}\left(w^{-1}, u, v; c, a, b\right)\right] \\
-\left(u^a - v^b\right)\left(u^a - w^c\right)\left(v^b - w^c\right) \\
\div (u-1)(v-1)(w-1)\left(u^a - v^b\right)\left(u^a - w^c\right)\left(v^b - w^c\right)
\end{gathered}$$

A nice result

$$\begin{aligned} L_P(t) &= \frac{abc}{6}\,t^3 + \frac{ab+ac+bc+1}{4}\,t^2 + (-\,\mathrm{s}(bc,a) - \mathrm{s}(ca,b) - \mathrm{s}(ab,c))\,t \\ &\quad + \left(\frac{3}{4} + \frac{a+b+c}{4} + \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + \frac{1}{abc}\right)\right) t + 1 \end{aligned}$$

some questions

what happens given a rational triangle?

should we generalize this application of Brion's theorem to n dimensions?