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Equations for Chow & Hilbert quotients w/ Angela Gibney

G alg. gp acting on proj. X.

Expect open set $U \subseteq X$ on which G acts freely U/G.

Problem Nice definition of X/G.

Our case $G = T^d \cong (\mathbb{C}^*)^d$ act on $X \subseteq \mathbb{P}^m$

e.g. $\mathbb{C}^*$ acts on $\mathbb{P}^3$ $t \cdot (x_0 : x_1 : x_2 : x_3)$
$= (x_0 : tx_1 : tx_2 : tx_3)$

Take $U = X \cap T^m$

What about X/T^d?
Want an orbit space : pts $\longleftrightarrow$ T^d-orbits

eg. $\mathbb{C}^* \curvearrowright \mathbb{C}^n$
$t(x_1, \dots, x_n) = (tx_1, \dots, tx_n)$

orbits: $\underline{0}$, PUNCTURED lines through $\underline{0}$

Standard answer : GIT (Geometric Invariant Theory)

(Throw away bad pts & form another space)

Philosophy :

X/T^d should be a moduli compactification of U/T^d

To do this, we'll embed U/T^d into a compact space P, & take the closure

Note For any $x \in U$, $\overline{T^d x} \subseteq \mathbb{P}^m$

has dim deg Hilbert polynomial

e.g. $\mathbb{C}^* \circlearrowright \mathbb{P}^3 \quad t \cdot x = (x_0 : t x_1 : t^2 x_2 : t^3 x_3) \qquad U = T^3$

$X = \mathbb{1} = (1:1:1:1) \quad T\mathbb{1} = \{(1:t:t^2:t^3) : t \in \mathbb{C}^*\}$

$\overline{T\mathbb{1}} = \{(1:t:t^2:t^3) : t \in \mathbb{C}^3\} \cup (1:0:0:0) \cup (0:0:0:1)$

$= V\langle x_0 x_3 - x_1 x_2,\ x_0 x_2 - x_1^2,\ x_1 x_3 - x_2^2 \rangle$

e.g. $x = (a:b:c:d) \quad a, b, c, d \neq 0$

$\overline{T^d x} = V\langle bc\, x_0 x_3 - ad\, x_1 x_2,\ b^2 x_0 x_2 - ac\, x_1^2,\ c^2 x_1 x_3 - bd\, x_2^2 \rangle$

For all such x, dim 1, deg 3, Hilbert poly $3t+1$

This means

$$U/T^d \hookrightarrow \mathrm{Hilb}_p(\mathbb{P}^m)$$
$$T^d x \longmapsto [\overline{T^d x}]$$

Def [Bialynicki-Birula/Sommese/Hu Kapranov] The Hilbert quotient $X /\!\!/^H T^d$ is the closure of $U/T^d \subseteq \mathrm{Hilb}_p(\mathbb{P}^m)$.

Def [Kapranov] The chow quotient $X /\!\!/^{ch} T^d$ is the closure of U/T^d in the Chow variety $\mathrm{Chow}_{d/\delta}(\mathbb{P}^m)$.

$\uparrow$ deg.

Goal: Given equations for $X \subseteq \mathbb{P}^m$. Describe equations for $X /\!/^* T^d$.

Q/ "give equations"?

A1/. Give equations in some (many) proj. embeddings.

A2/. $X /\!/^*_{T^d} \hookrightarrow \mathbb{P}^m /\!/^* T^d$

$\mathbb{P}^m /\!/^* T^d$ is a toric variety. [KSZ]

Give equations for $X /\!/^* T^d$ in the Cox-ring of $\mathbb{P}^m /\!/^* T^d$.

Toric varieties are generalizations of proj spaces

The Cox ring (homogeneous coordinate ring) of X_Σ is a polynomial ring $\mathbb{C}[x_1, \dots, x_r]$ where

(certain) ideals $\rightsquigarrow$ subschemes of X_Σ

e.g. $X_\Sigma = \mathbb{P}^m$ homogeneous saturated ideals $\longleftrightarrow$ subschemes of $\mathbb{P}^m$.

Thm [Gibney–M]
Let X_Σ be a toric subvariety of $\mathbb{P}^m /\!/^* T^d$ with

$$X /\!/^*_{T^d} \subseteq X_\Sigma \subseteq \mathbb{P}^m /\!/^* T^d$$

Then the ideal I of $X /\!/^* T^d$ inside $\mathrm{Cox}(X_\Sigma)$ is contained by homogenizing

$$I(X) \subseteq \mathbb{C}[x_0, \dots, x_n]$$

inside $\mathrm{Cox}(X_\Sigma) = \mathbb{C}[x_0, \dots, x_m, y_1, \dots, y_s]$

This gives a GIT constructions of $X /\!/^* T^d$, and all other GIT quotients.

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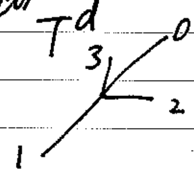
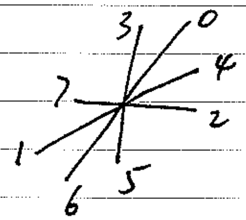
$X /\!\!/_\alpha T^d$ can be obtained from $X /\!\!/^* T^d$ by VGIT.

e.g. $\mathbb{C}^* \curvearrowright \mathbb{P}^3$ $\qquad \mathbb{P}^3 /\!\!/^M T^d \qquad \mathbb{C}[x_0, x_1, x_2, x_3, y_4, y_5, y_6, y_7]$

$\mathbb{P}^3 /\!\!/^{ch} T^d$





$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 & & & & \\ 0 & 0 & 0 & 1 & & I & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & & & \end{array}\right)$$

e.g. $X = V\langle x_0x_3 - x_1x_2\rangle$

$t.x = (x_0 : tx_1 : t^2x_2 : t^3x_3)$

$I = \langle x_0x_3y_5 - x_1x_2y_6y_7\rangle$

$X /\!\!/^M T = V(I) /\!\!/_\alpha \tilde{T}$ ← 6-dim torics

$= \mathbb{P}^1$

Main application / Motivation

$\overline{M_{0,n}}$ = moduli space of stable n pted genus ~~curves~~ zero

This is a compactification of $M_{0,n}$ = "stable" ⟶ "smooth"

i.e. $M_{0,n}$ is all ways to arrange n distinct pts on $\mathbb{P}^1 / \mathrm{aut}(\mathbb{P}^1)$

e.g. $M_{0,3} = pt$

e.g. $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 5 \end{pmatrix}$$

Taking n distinct pts in $\mathbb{P}^1 \rightsquigarrow$ $2\times n$ matrix

pt in $G(2,n)$

Note: 1) $PGL(2)$ action doesn't change the pt.

2) pts distinct $\Rightarrow$ all 2×2 minors nonzero.

$$\begin{pmatrix} 2 & 3 & 0 & 5 \\ 0 & 3 & 4 & 25 \end{pmatrix}$$

3) Choose different reps in $\mathbb{P}^1 \longrightarrow T^n$ action on $G(2,n)$.

$PGL(2)$-orbits of distinct pts in $\mathbb{P}^1$ $\longleftrightarrow$ T^n orbits of pts in $G(2,n)$ with nonvanishing Plücker coords.

$$M_{0,n} = G^\circ(2,n)/T^n$$

$x_{ij}x_{kl} - x_{ik}x_{jl} - x_{il}x_{jk}$

$$X = G(2,n) \cap T^{\binom{n}{2}-1}$$

(Recall: Plücker embedding $G(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$

Thm [Kapranov]

$$\overline{M_{0,n}} = G(2,n) /\!/^{ch} T^n \hookrightarrow \mathbb{P}^{\binom{n}{2}-1} /\!/^{ch} T^n$$

$$= G(2,n) /\!/^{H} T^n \hookrightarrow \mathbb{P}^{\binom{n}{2}-1} /\!/^{H} T^n$$

We can now apply our theorem except $\mathbb{P}$ are hard to get your hands on

Luckly we understand X_Δ (Δ = space of phylogenetic reps) y.t.

well understand lots of beautiful combinatorics

$$\overline{M}_{0,n} \longrightarrow X_\Delta \hookrightarrow \mathbb{P}^m /\!/^{ch} T^n$$

$$X_\Delta \hookrightarrow \mathbb{P}^n /\!/^{H} T^n$$

Coxring of X_Δ

variables $\longleftrightarrow$ bdy divisors on $\overline{M_{0,n}}$.