General LLT Polynomials

CRM Workshop on Interactions between Algebraic Combinatorics and Algebraic Geometry

Mark Haiman

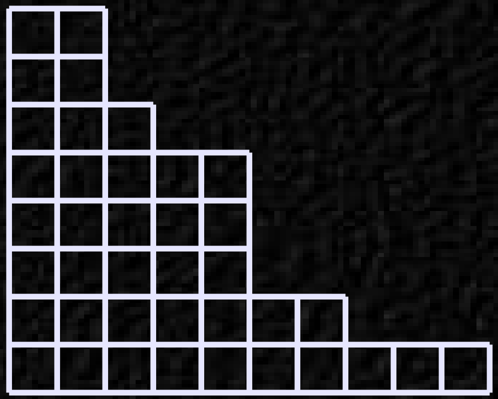
joint work with

Ian Grojnowski

Part I

The combinatorial polynomials of

Lascoux,
Leclerc
&
Thibon



The k-core and k-quotient of a partition

$(k = 4)$

General LLT Polynomials

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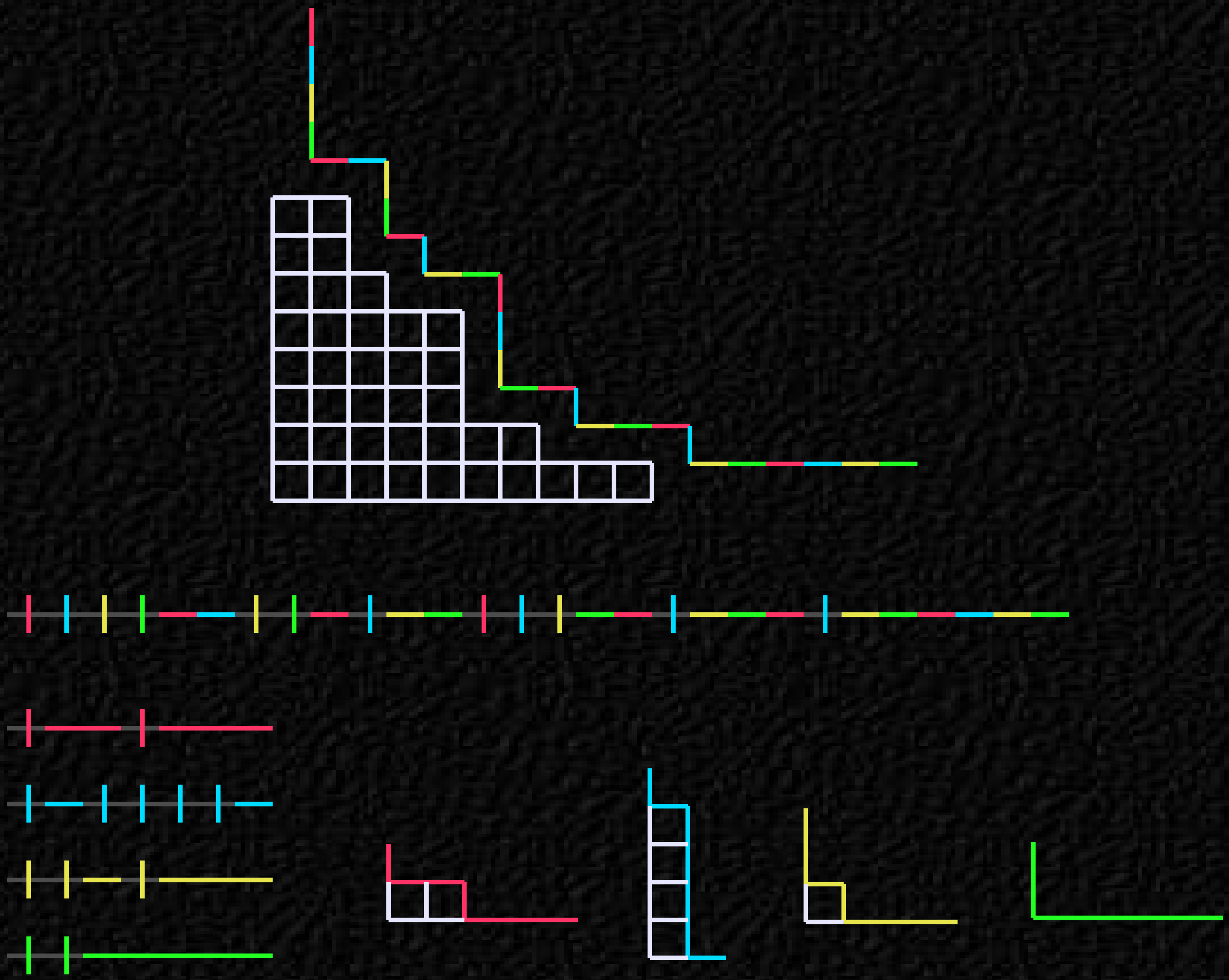
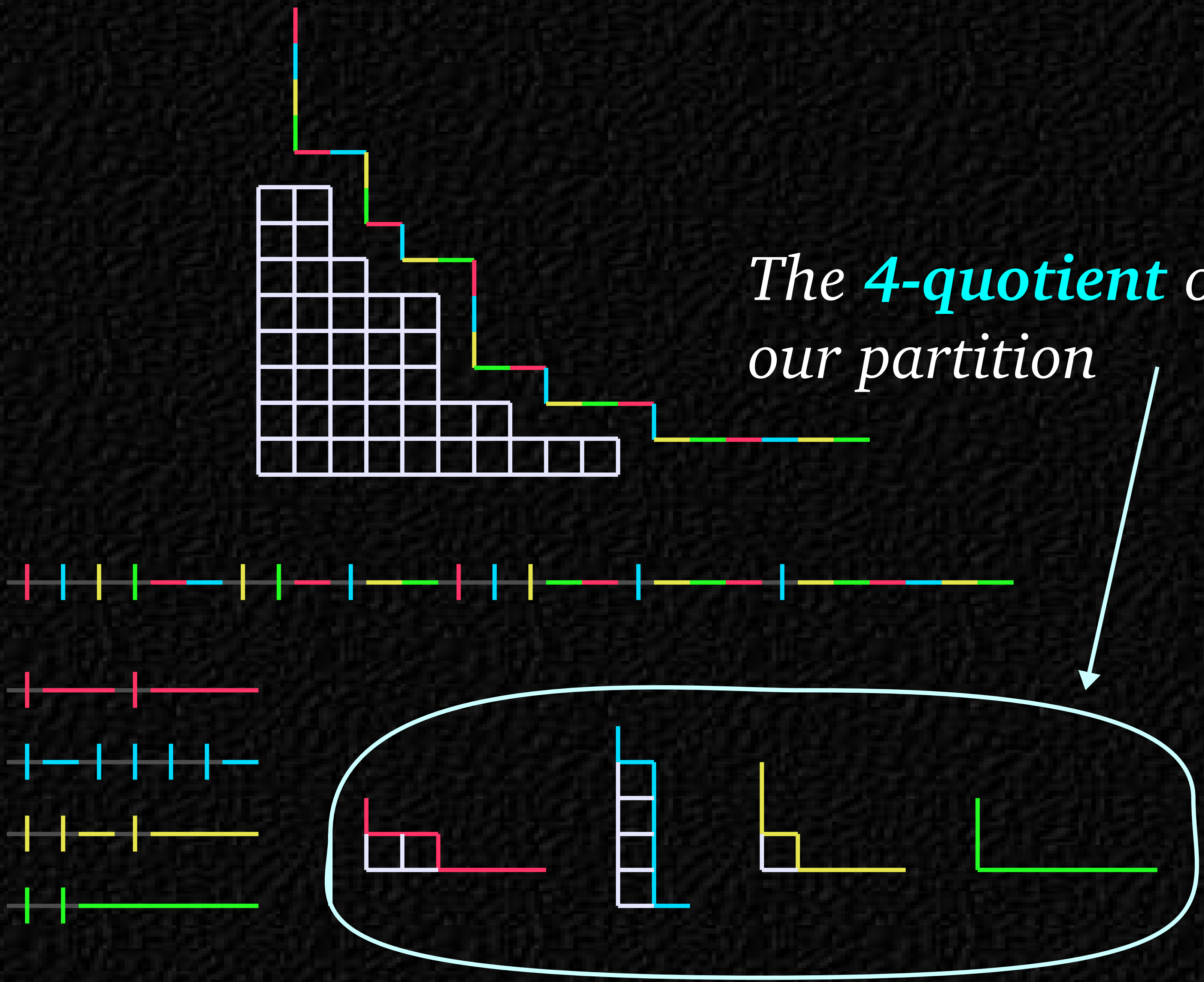
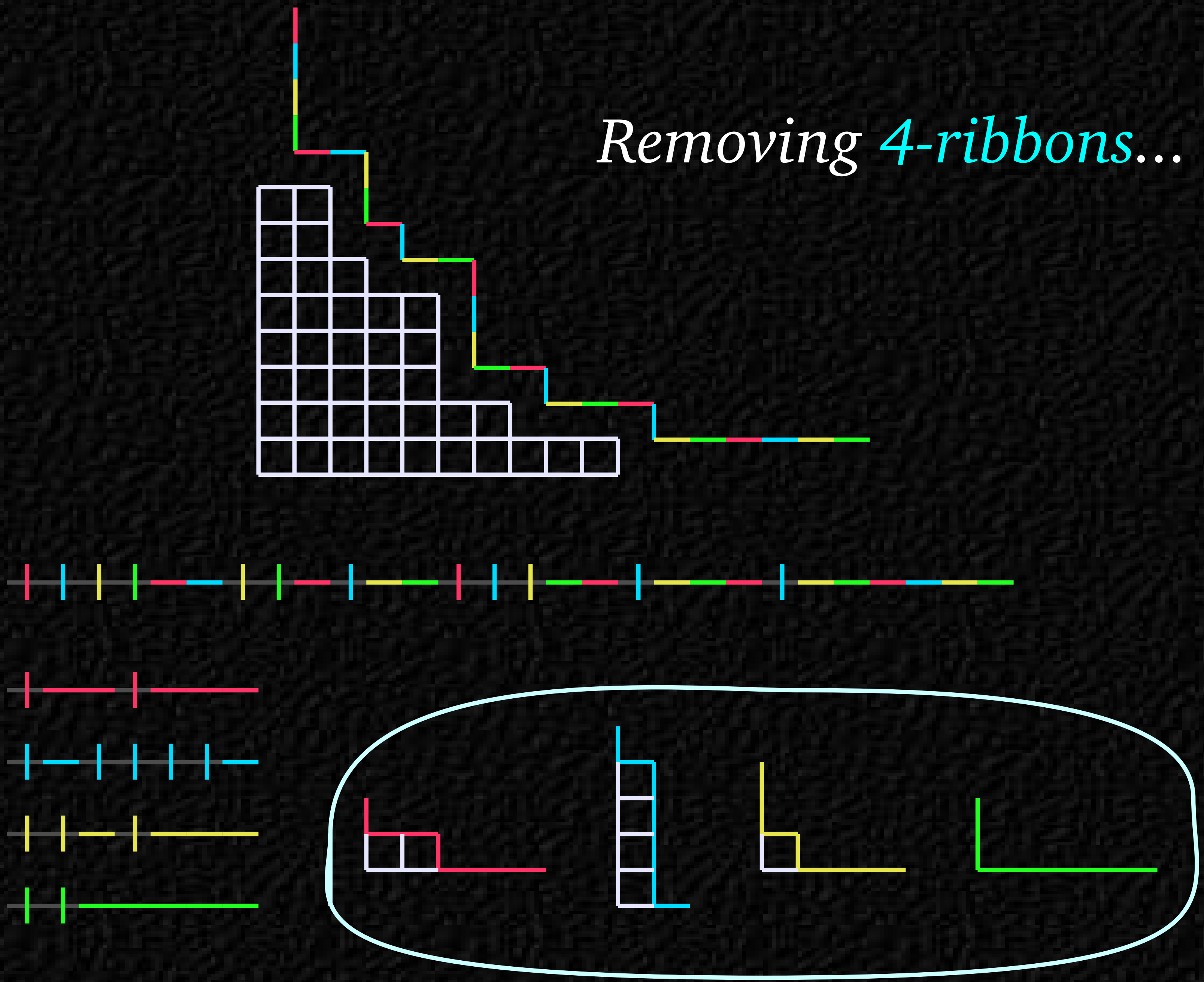



The **4-quotient** of our partition

Removing 4-ribbons…



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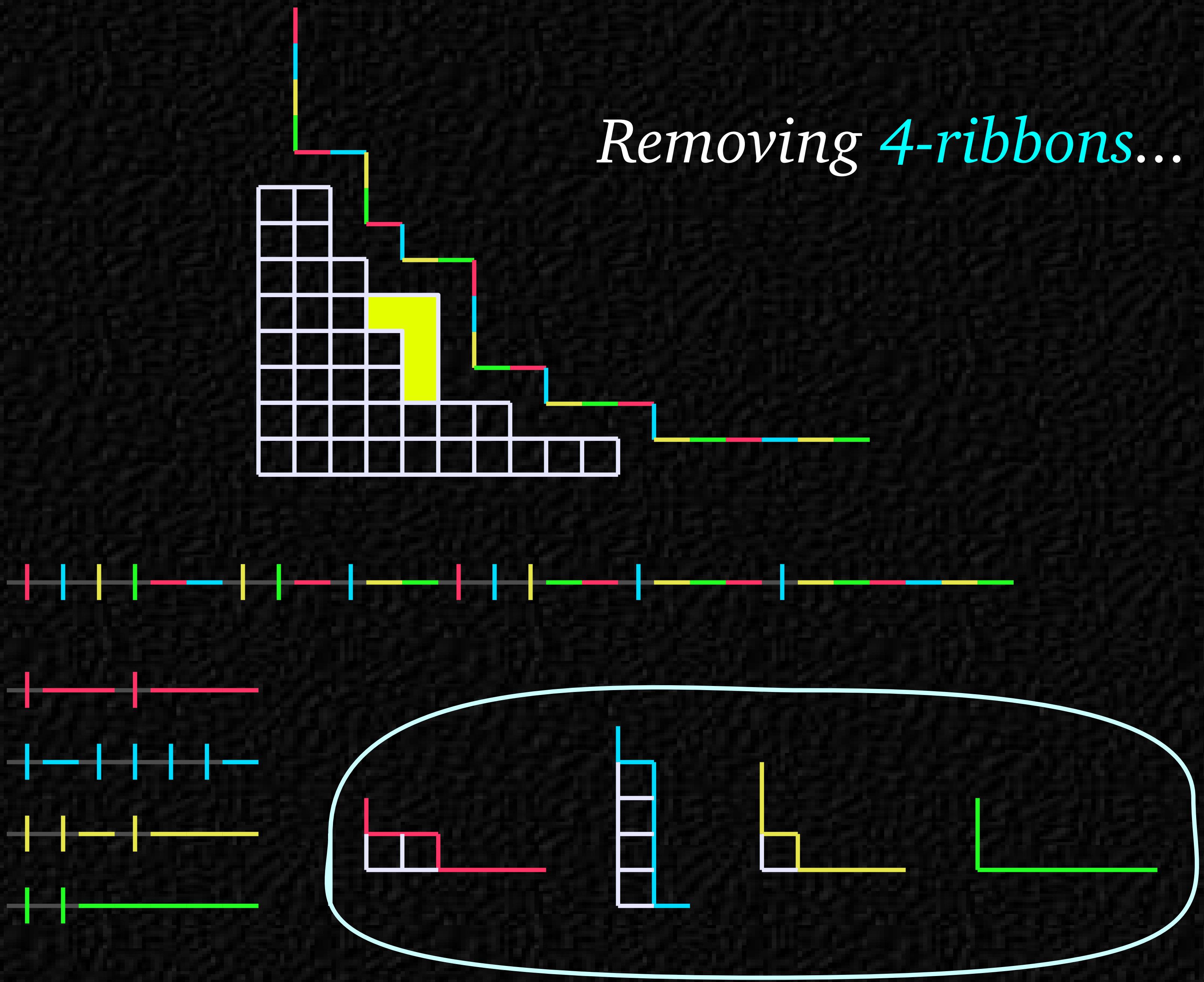

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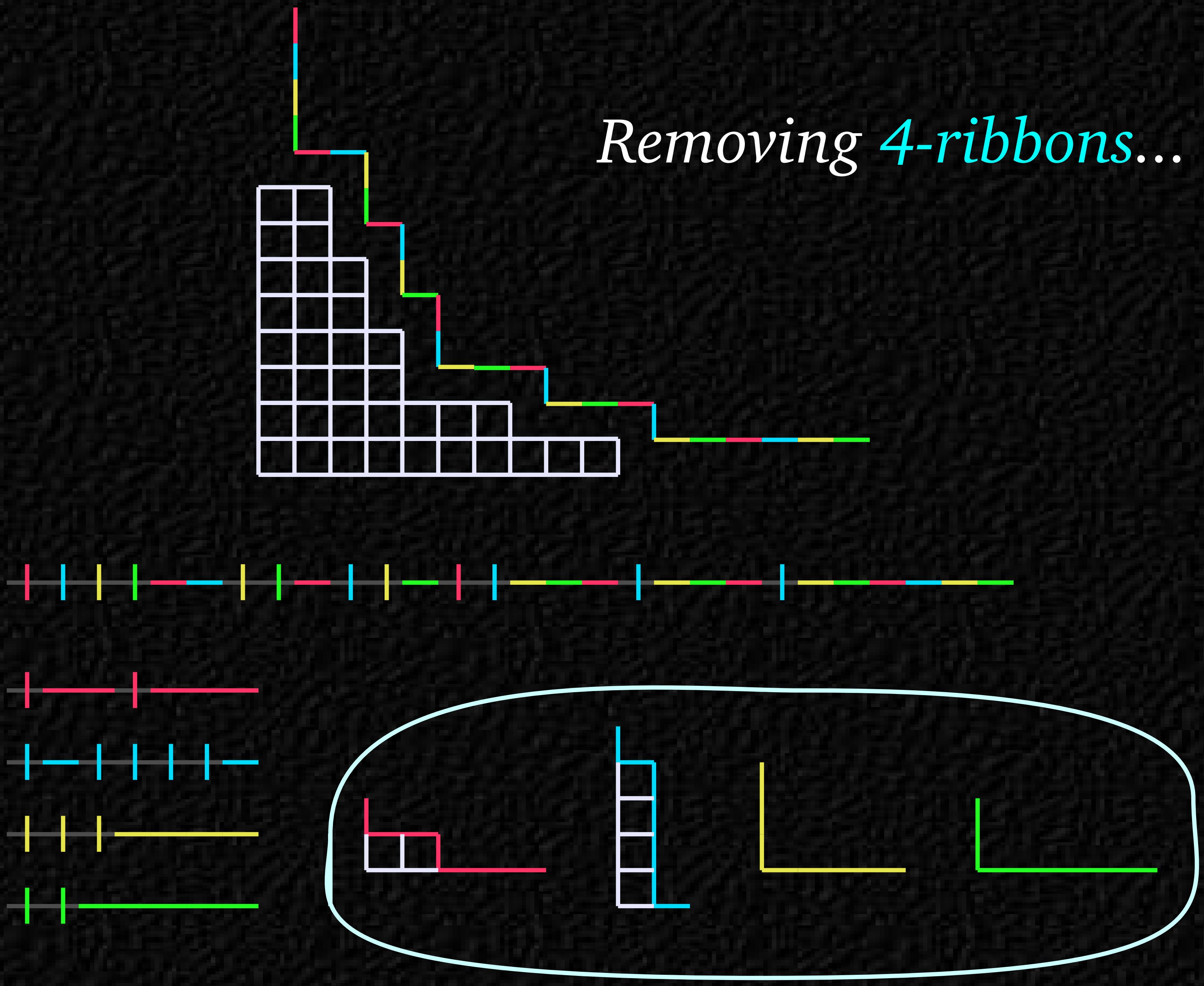

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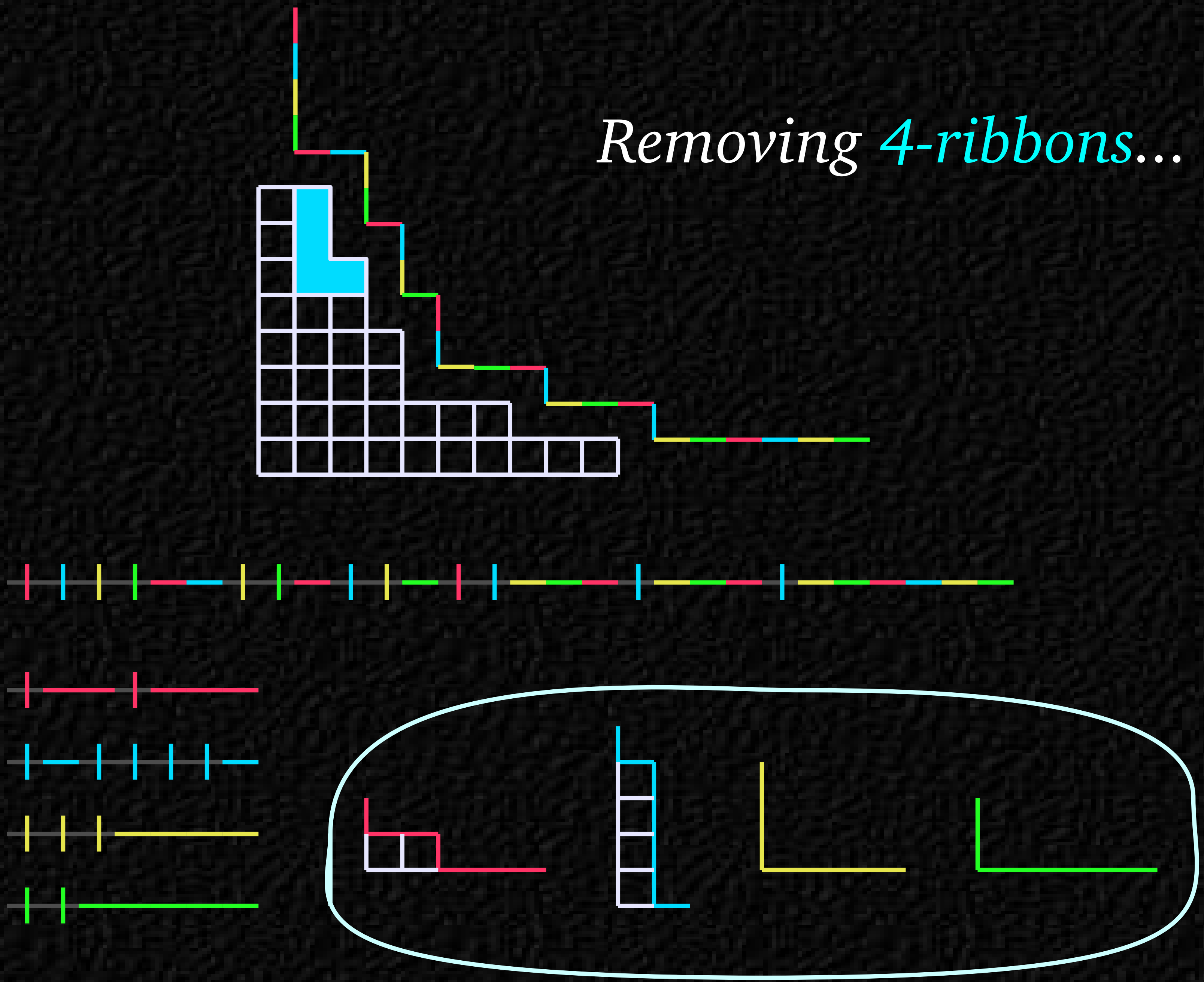

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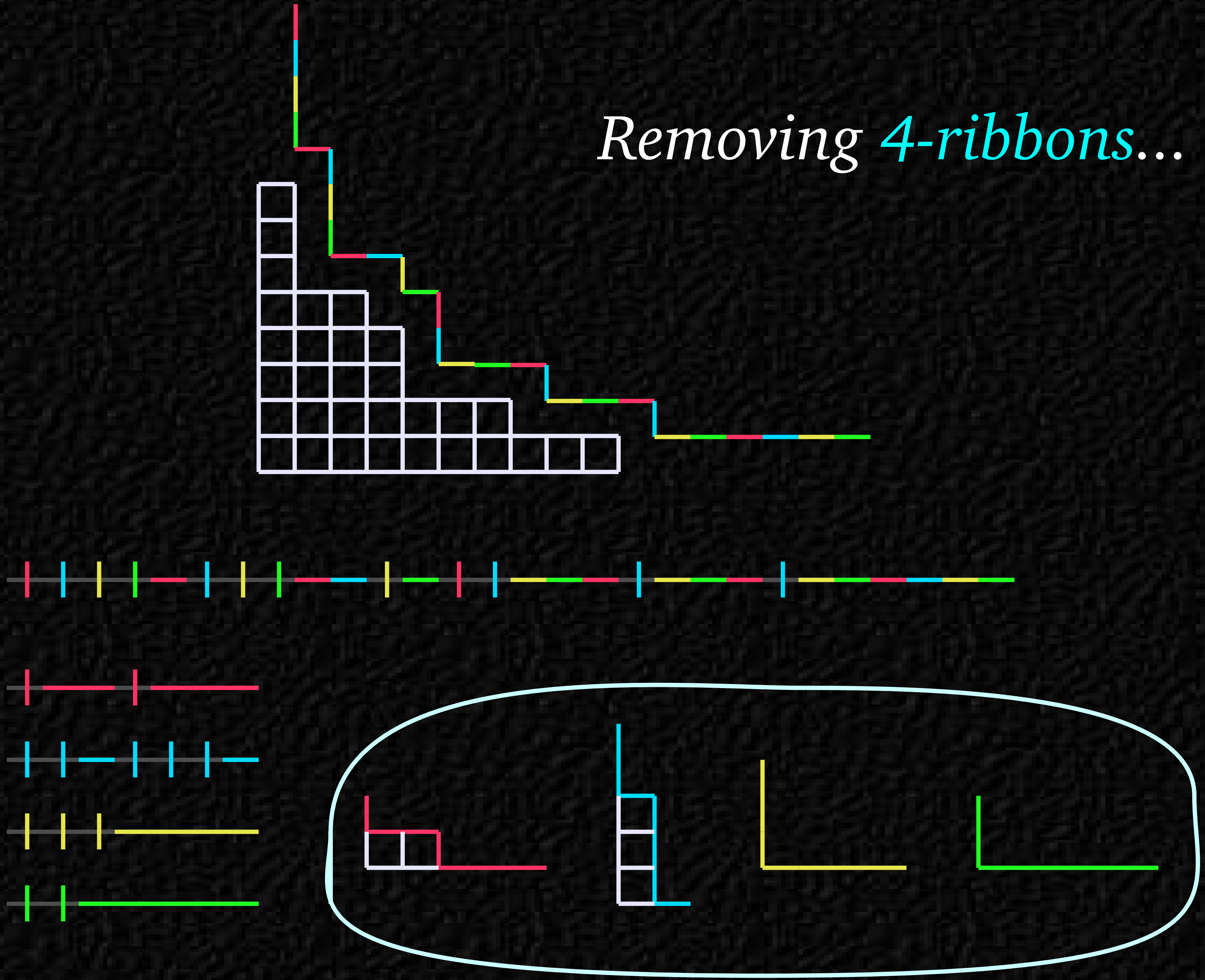

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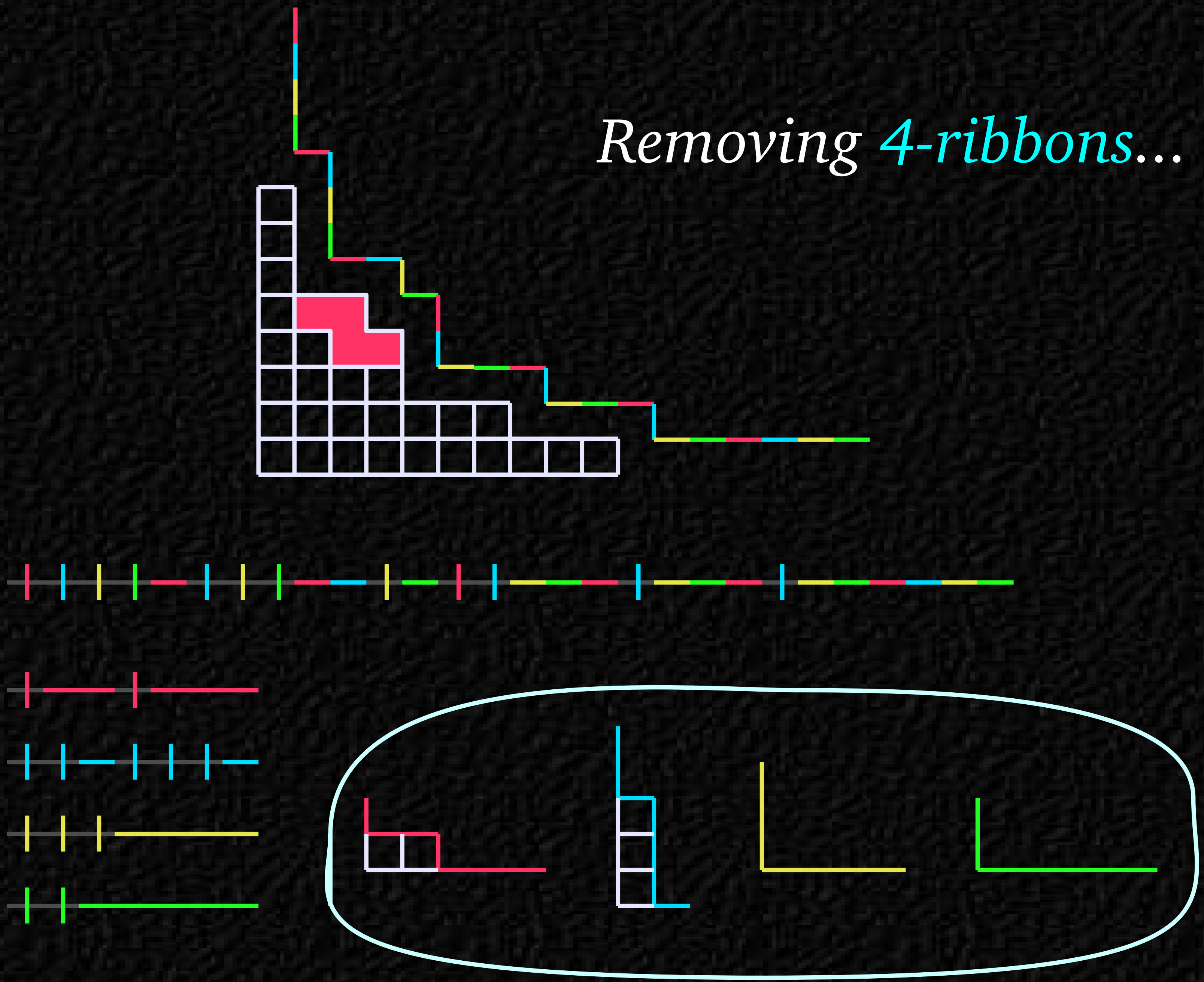

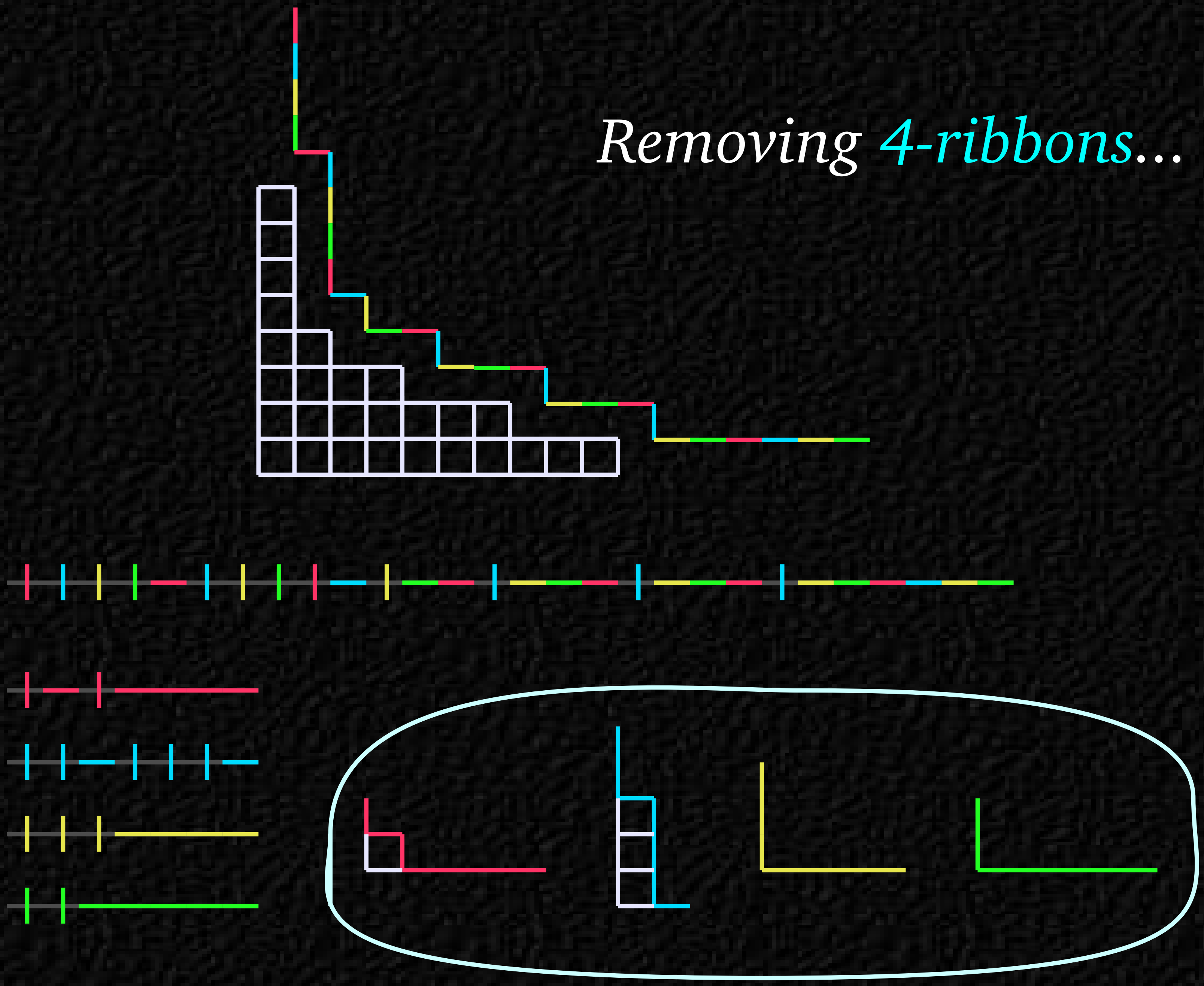
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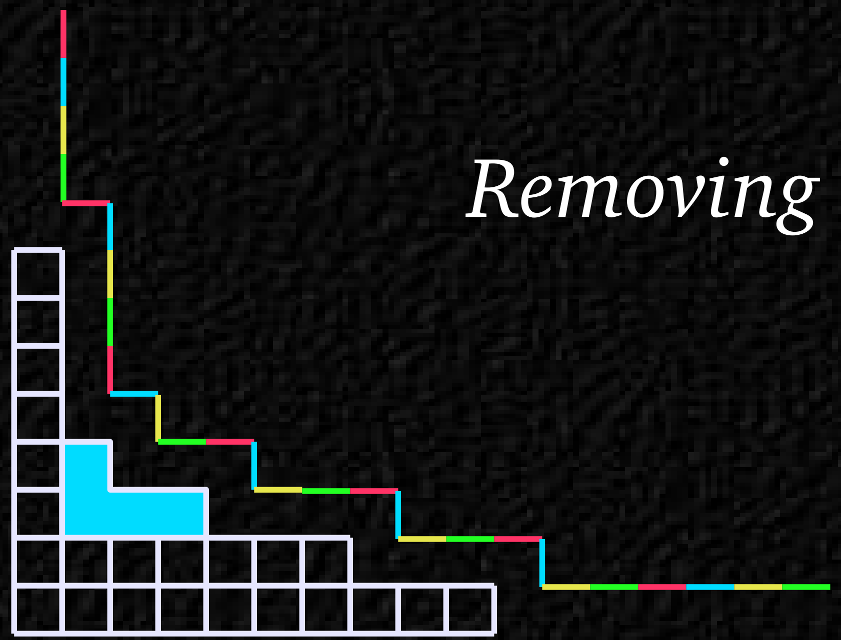

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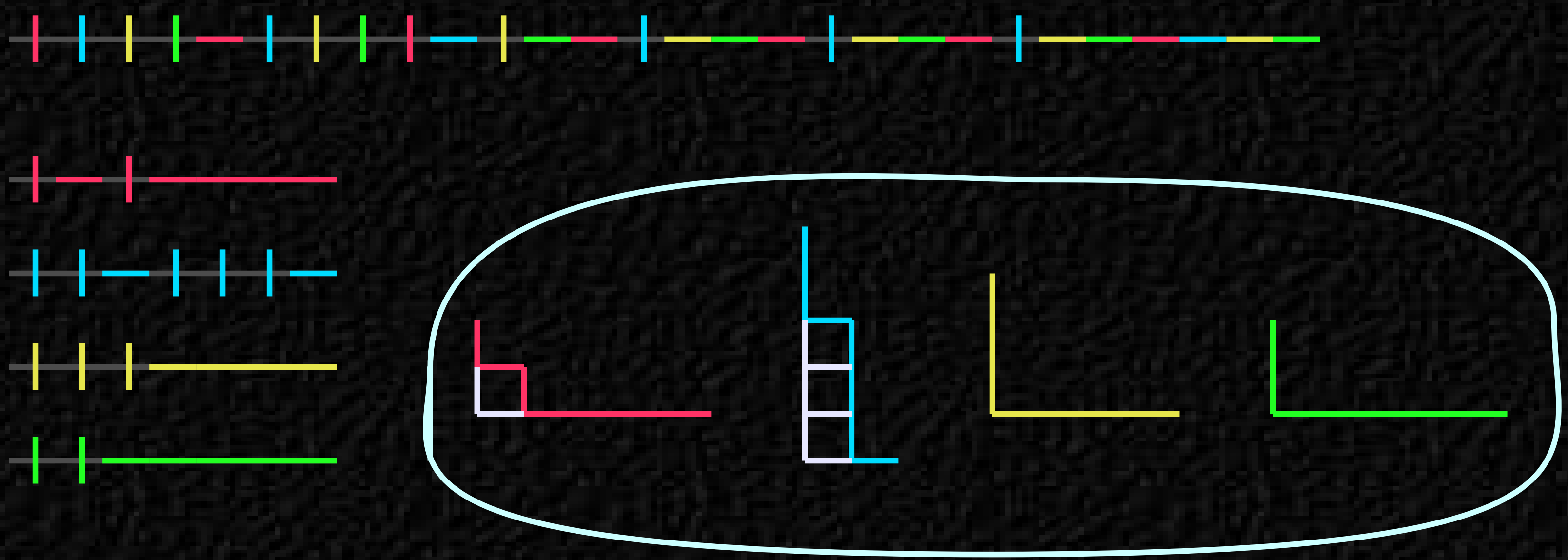

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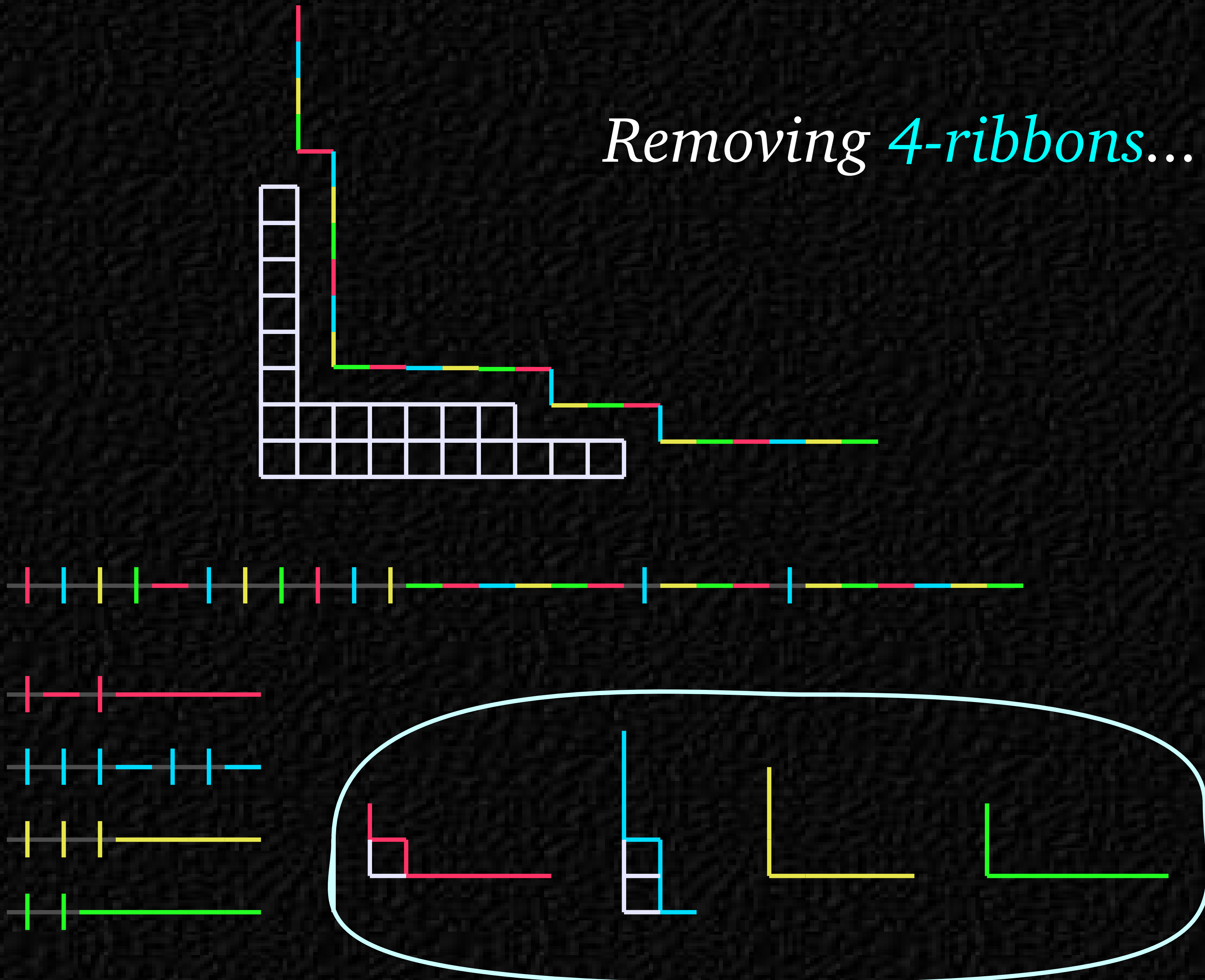

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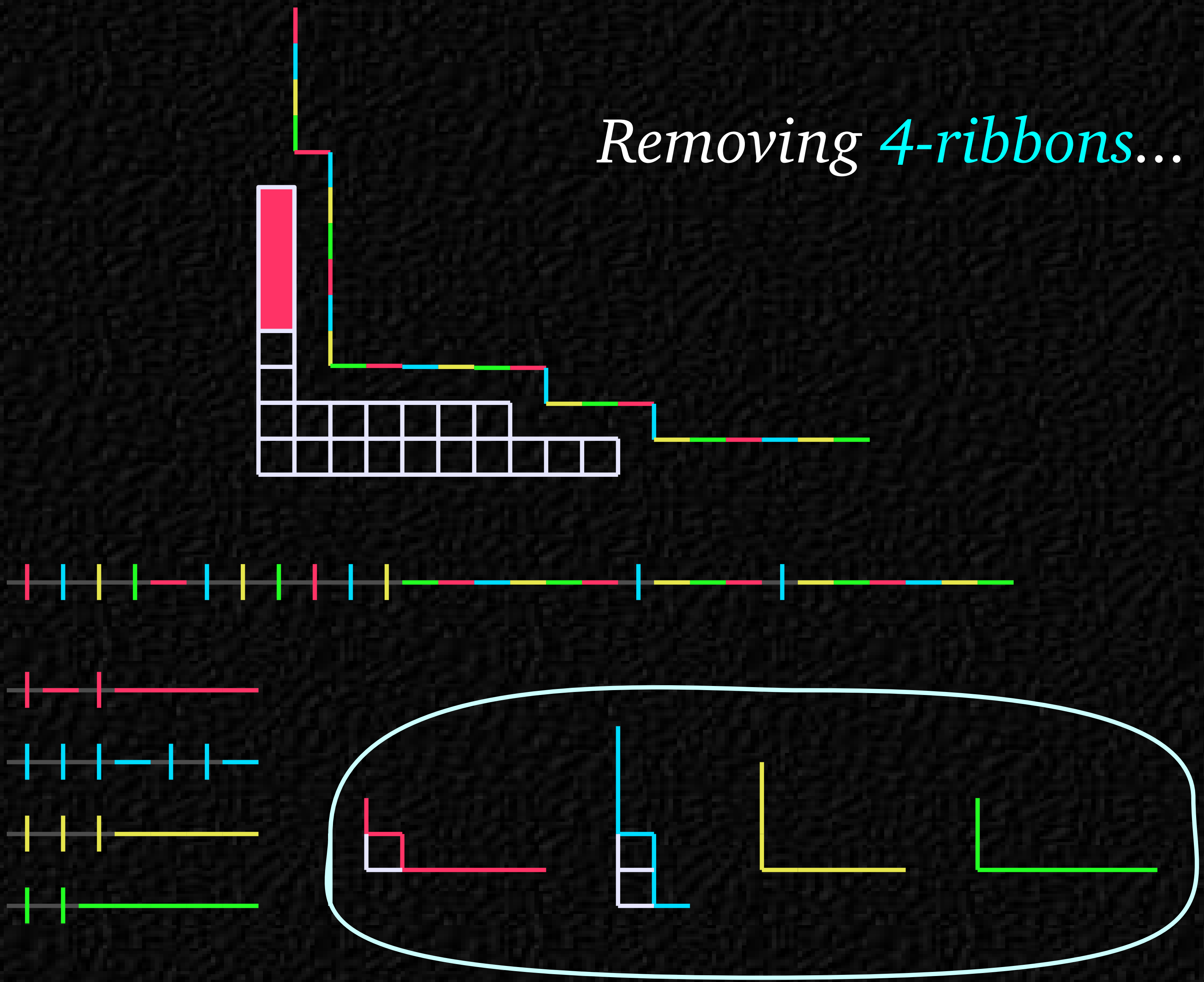

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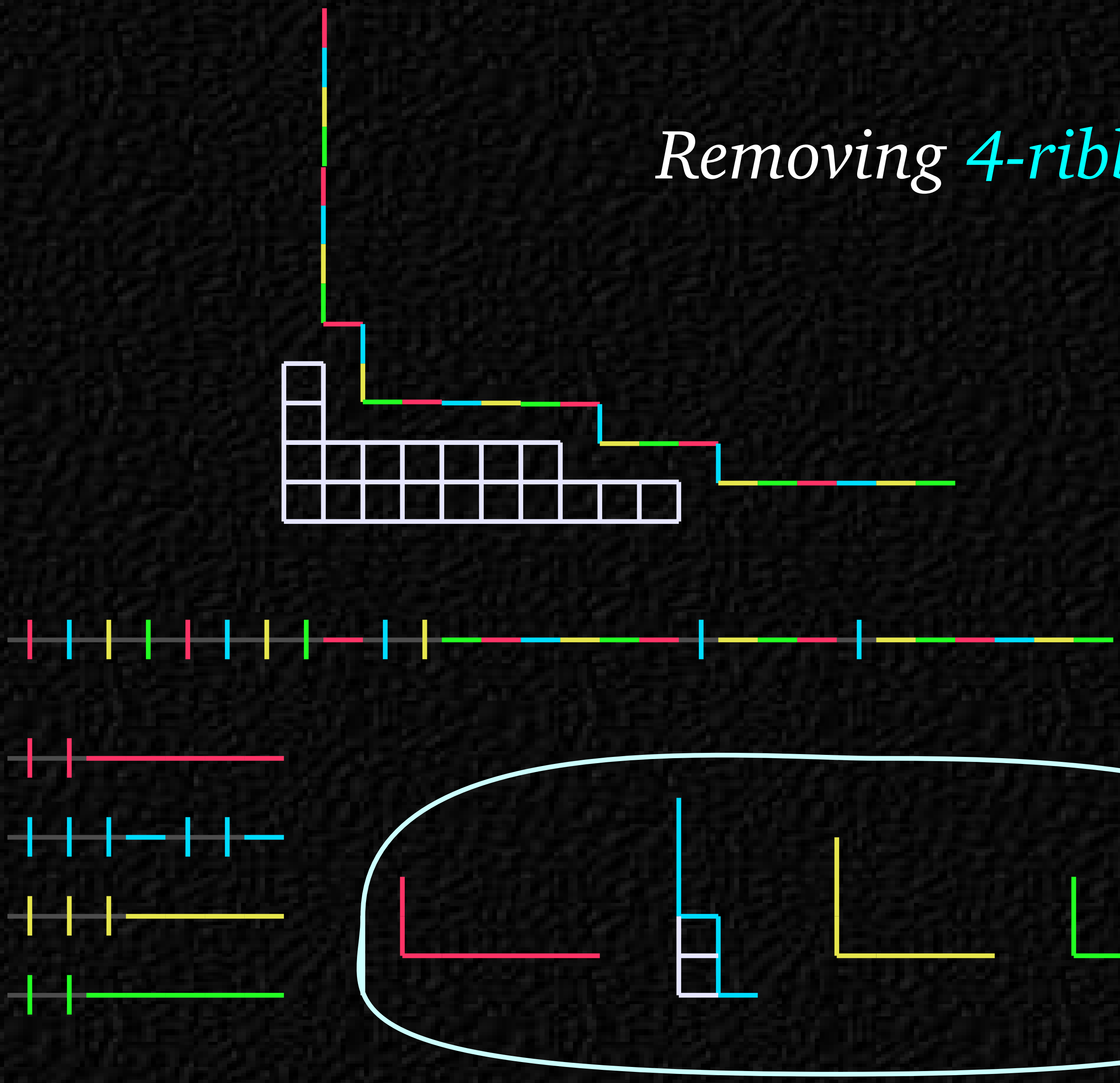

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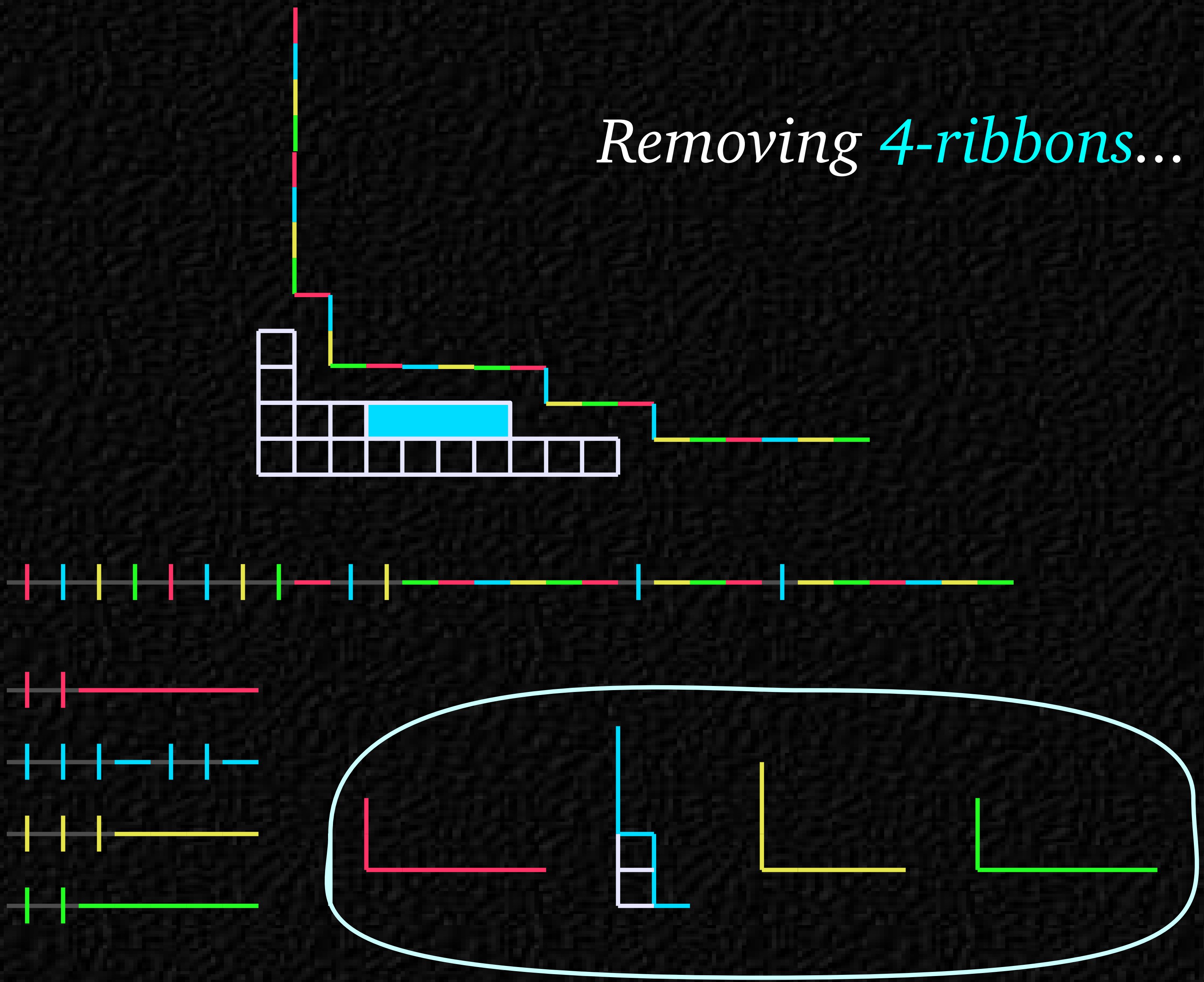

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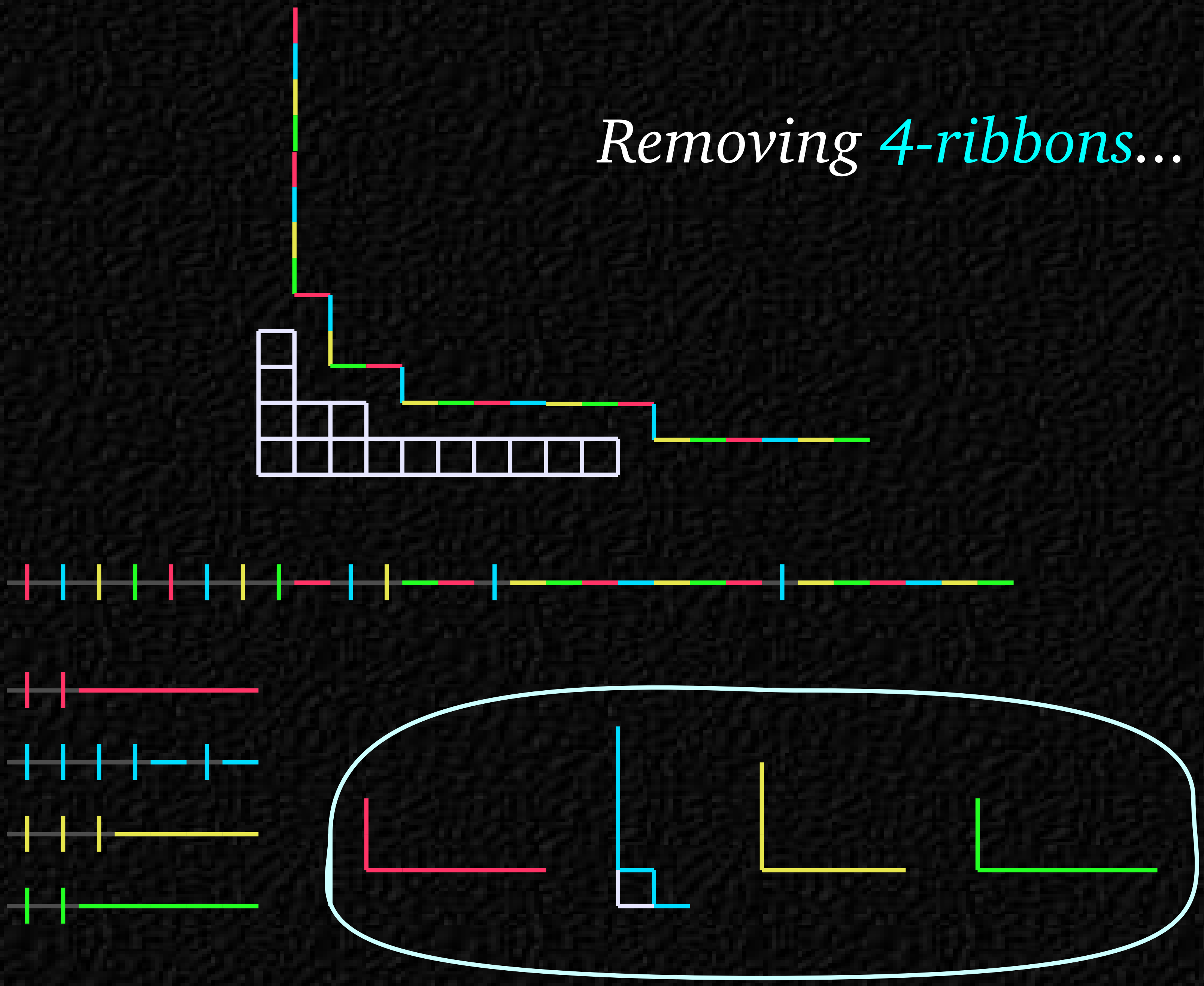

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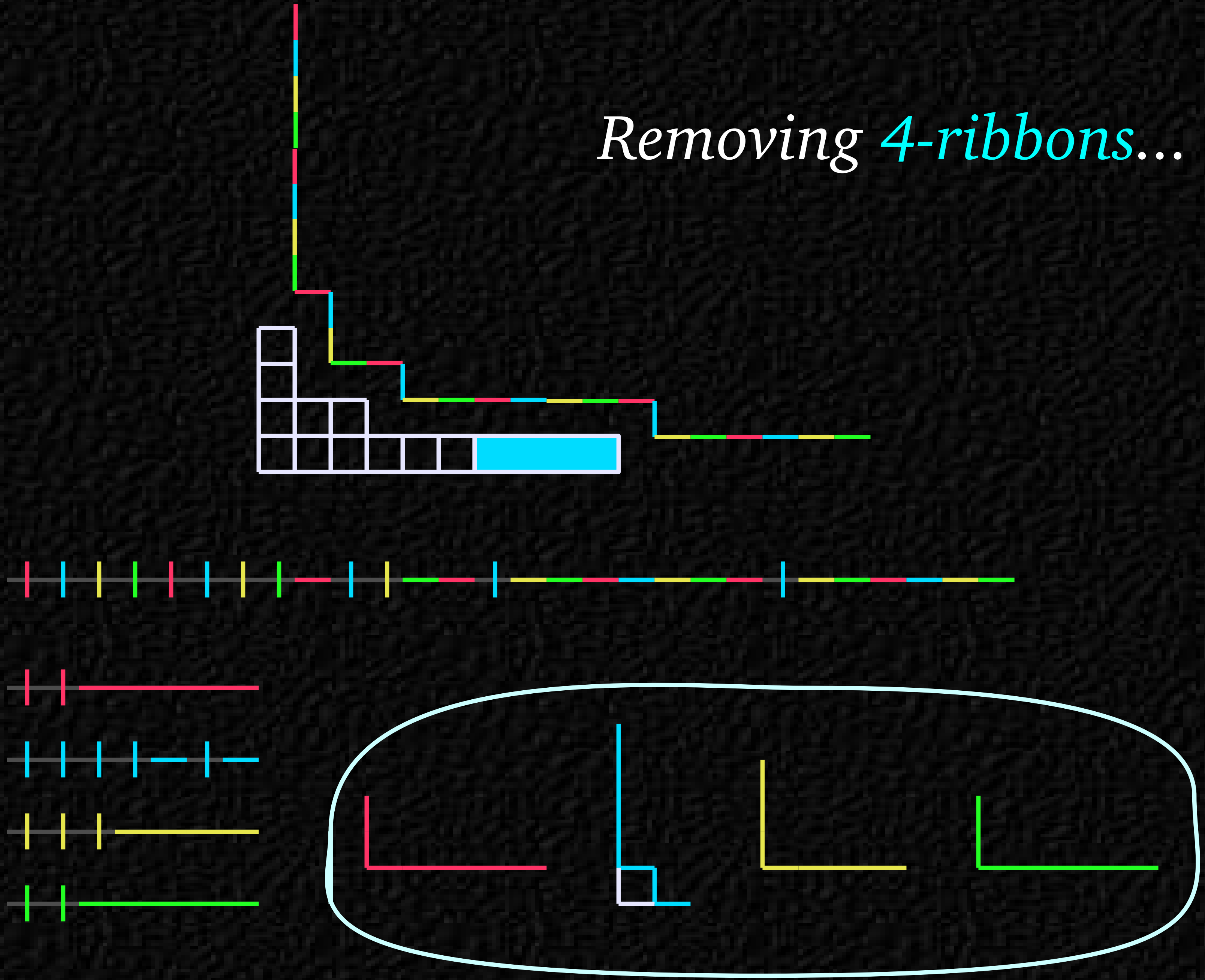

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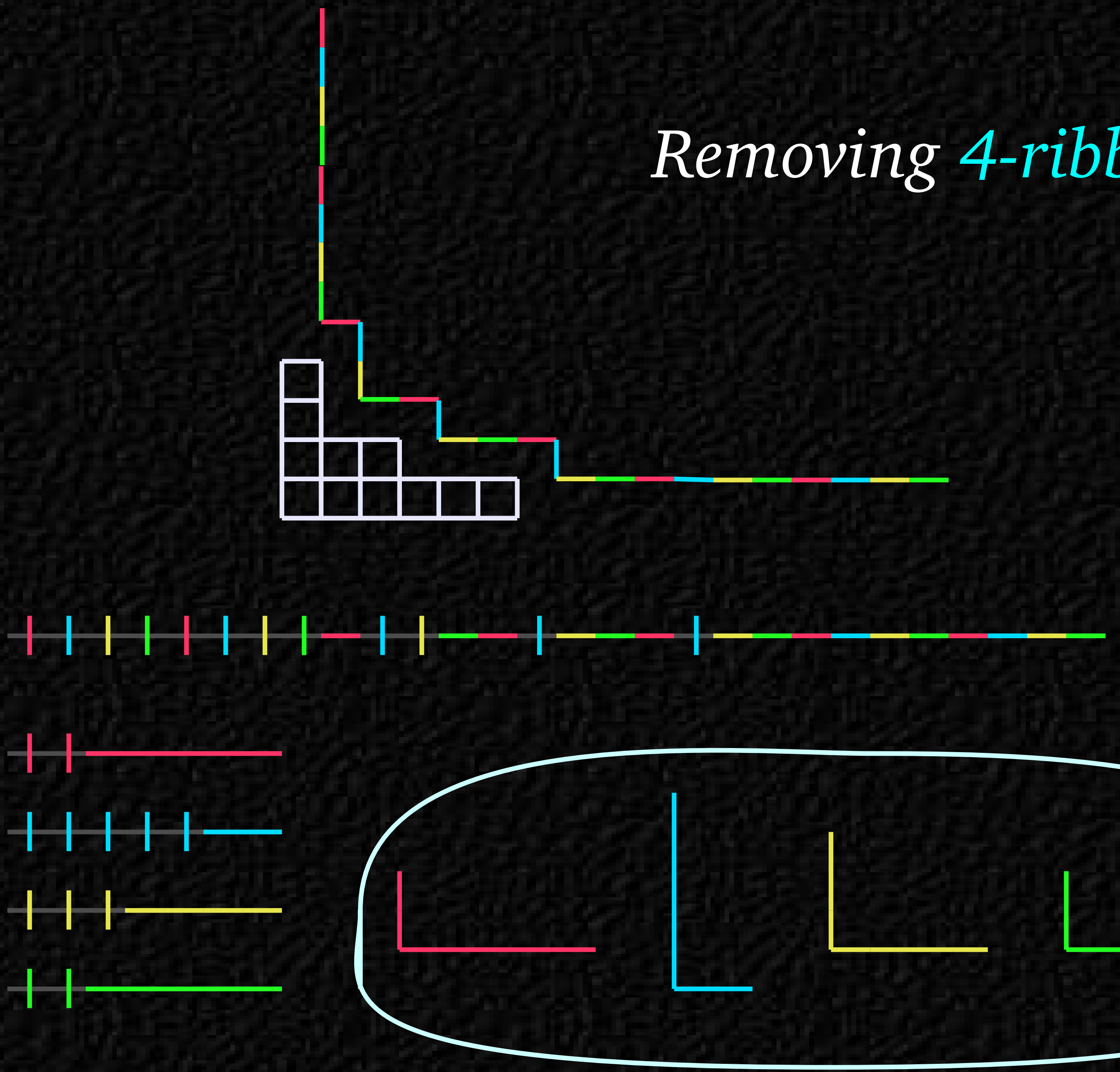
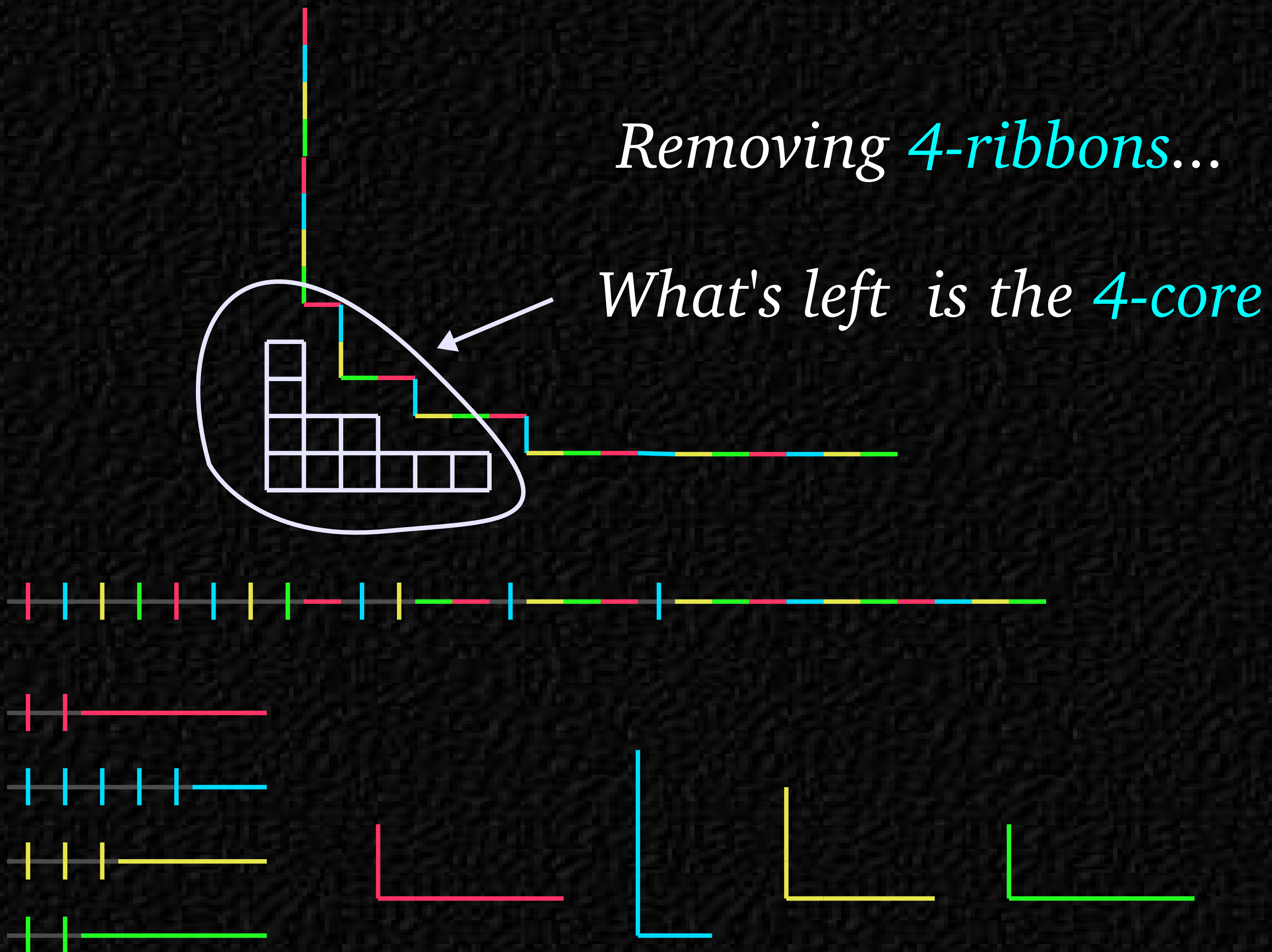
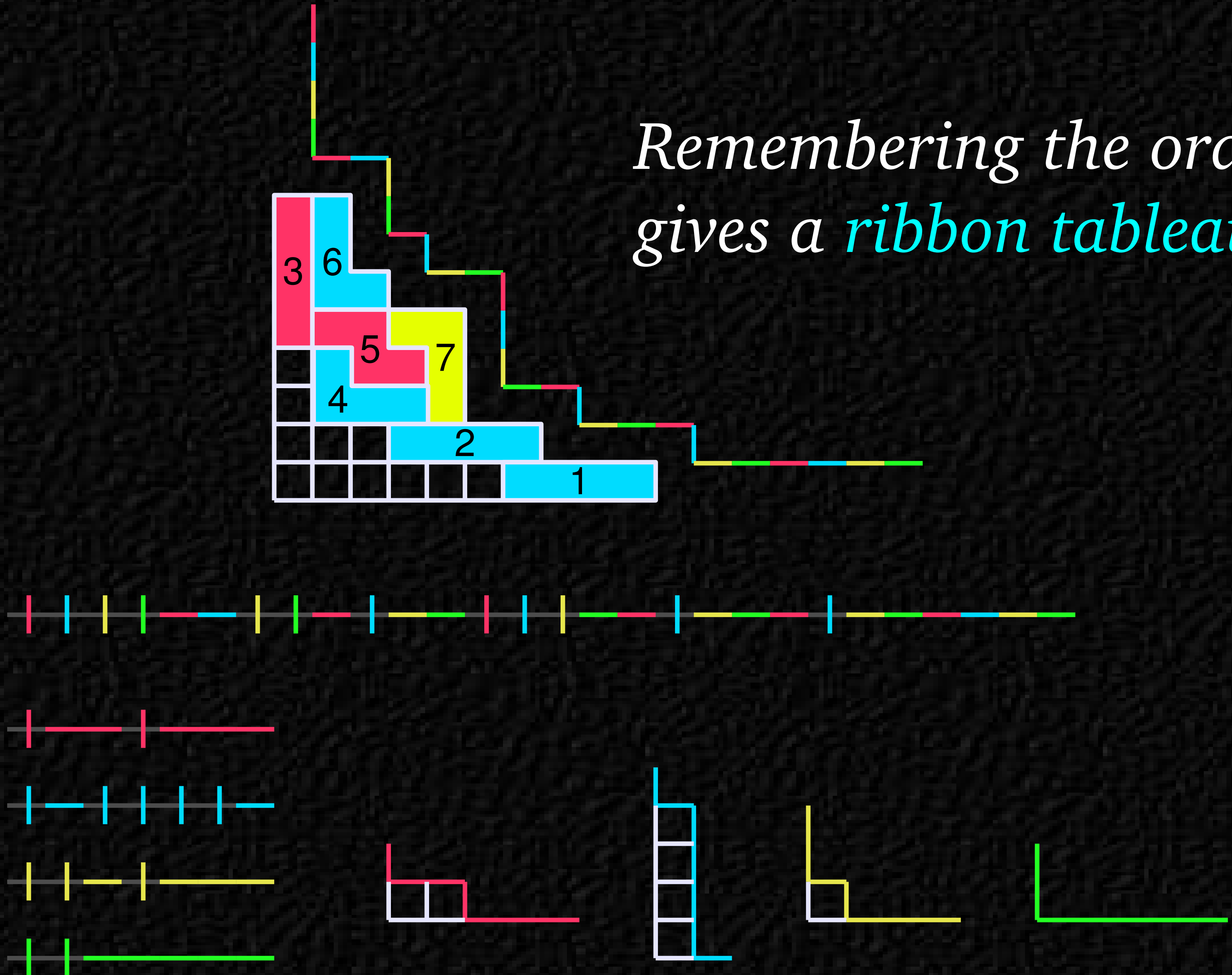
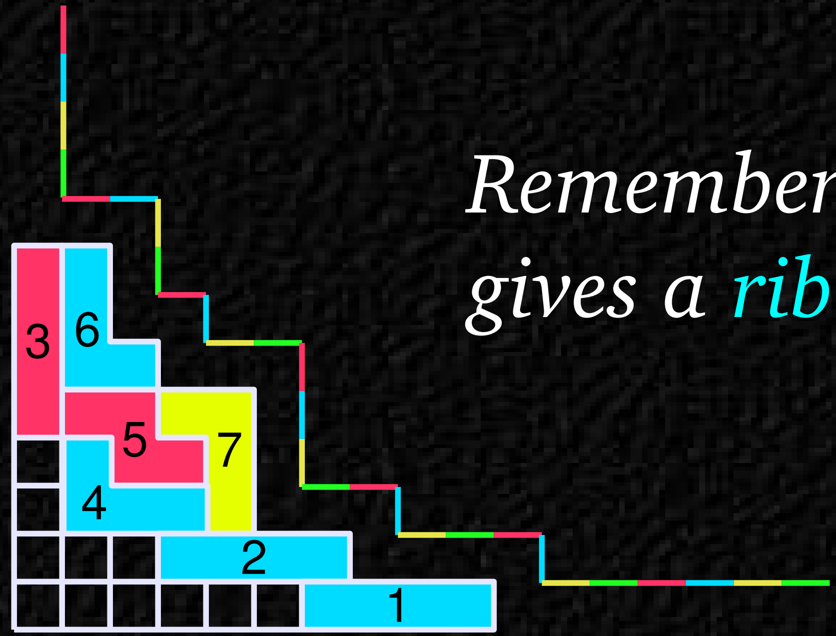



Removing 4-ribbons...
What's left is the 4-core

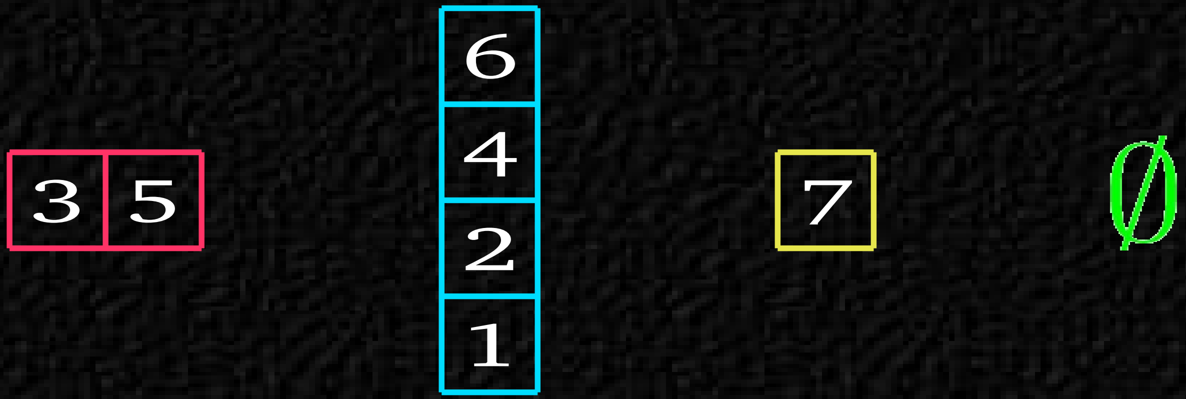
Remembering the order gives a ribbon tableau…

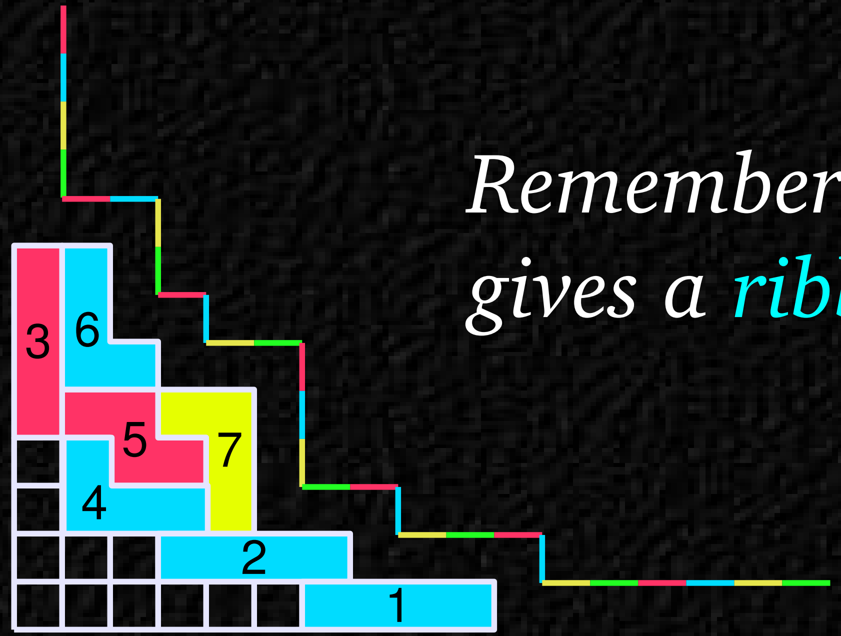




Remembering the order gives a ribbon tableau...

...and corresponding standard tableau on the k-quotient.





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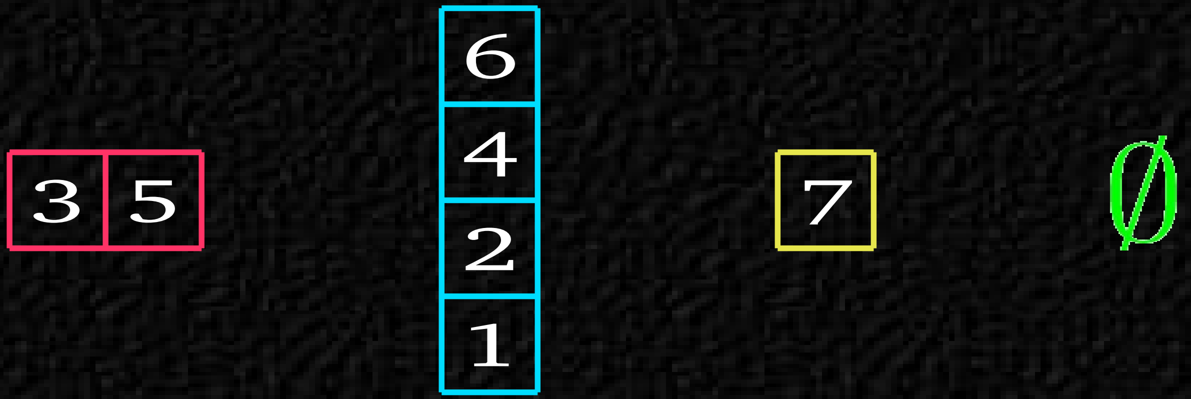
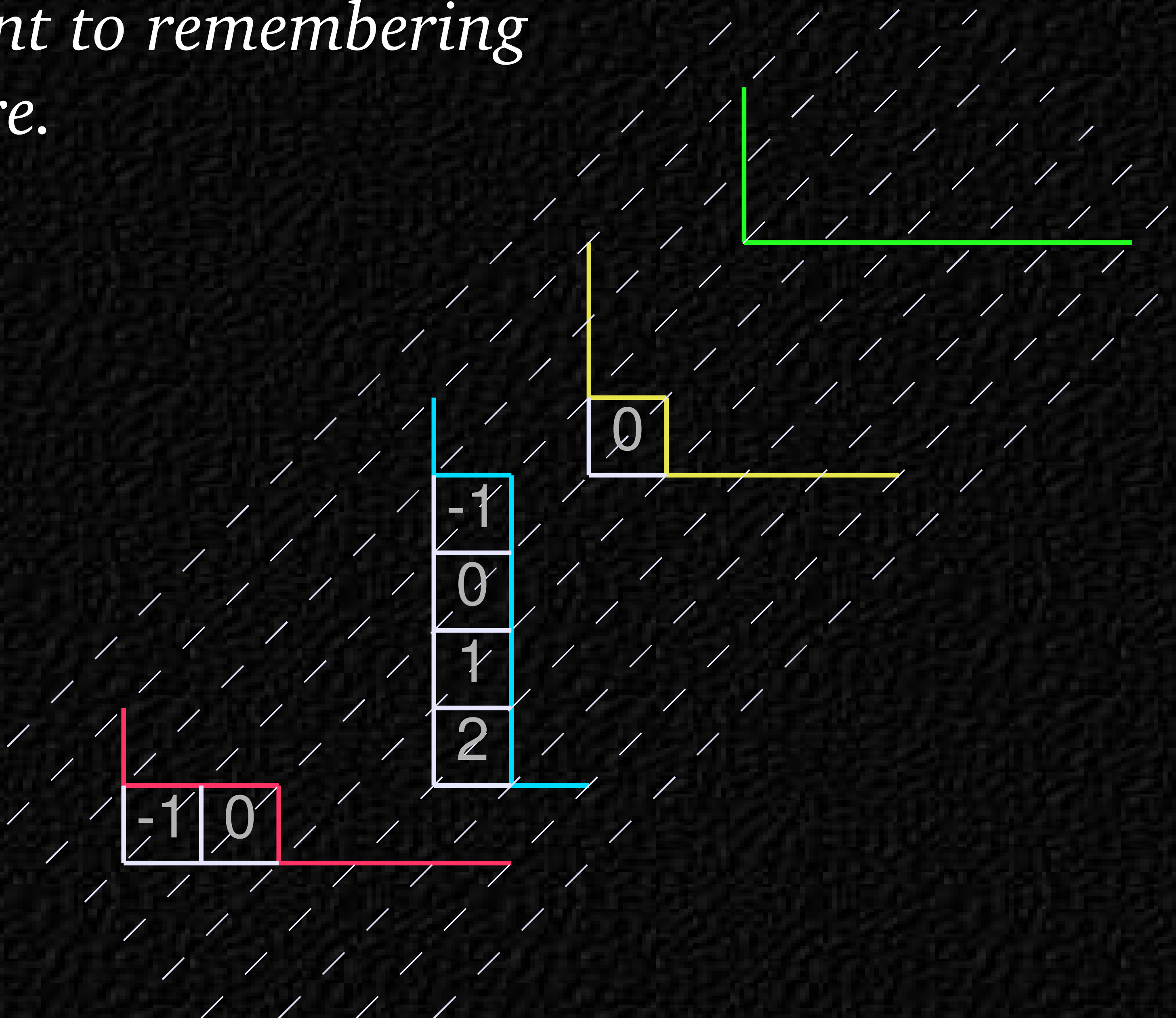
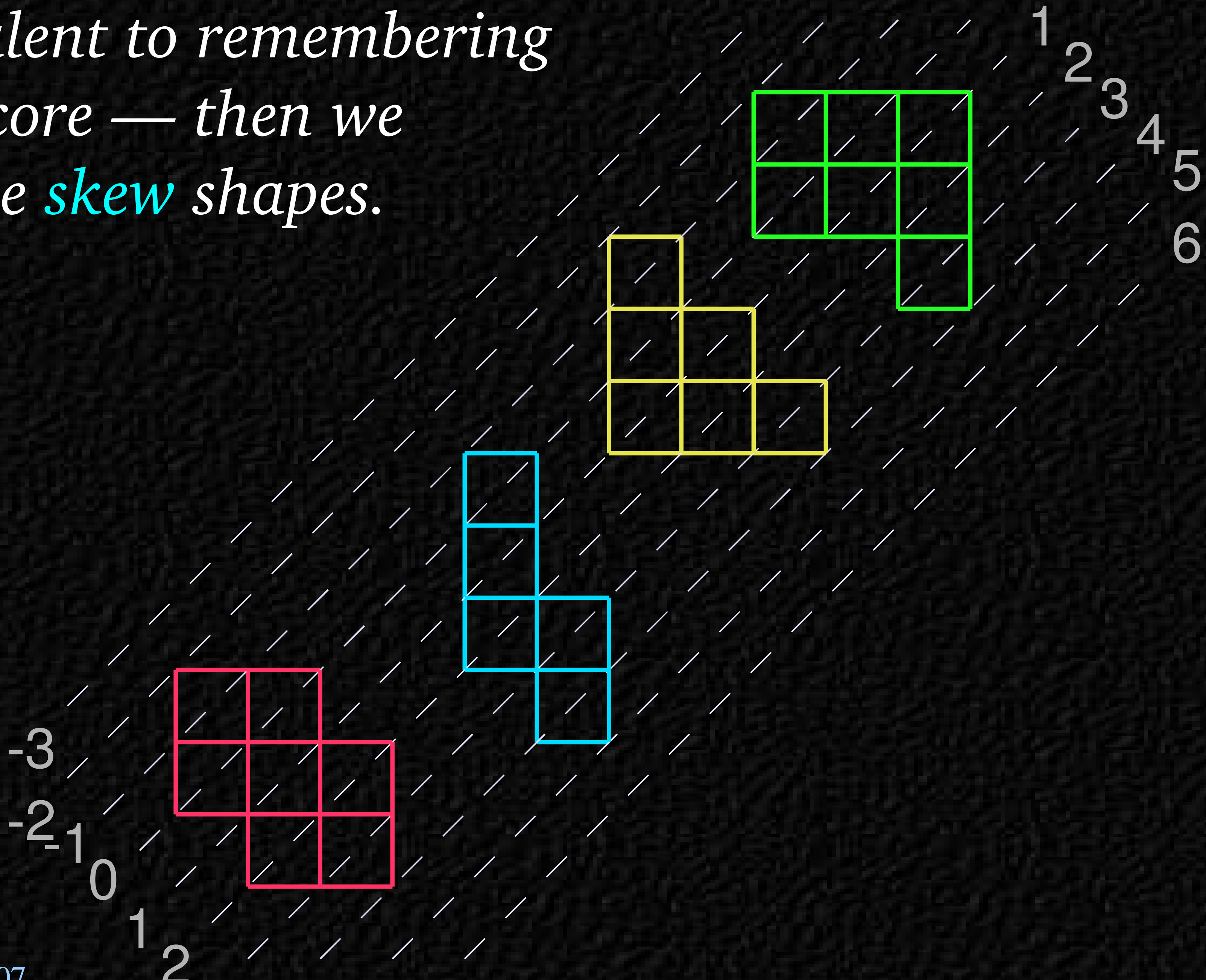


This also works for semistandard tableaux.

Aligning content lines in the k-quotient is equivalent to remembering the k-core.

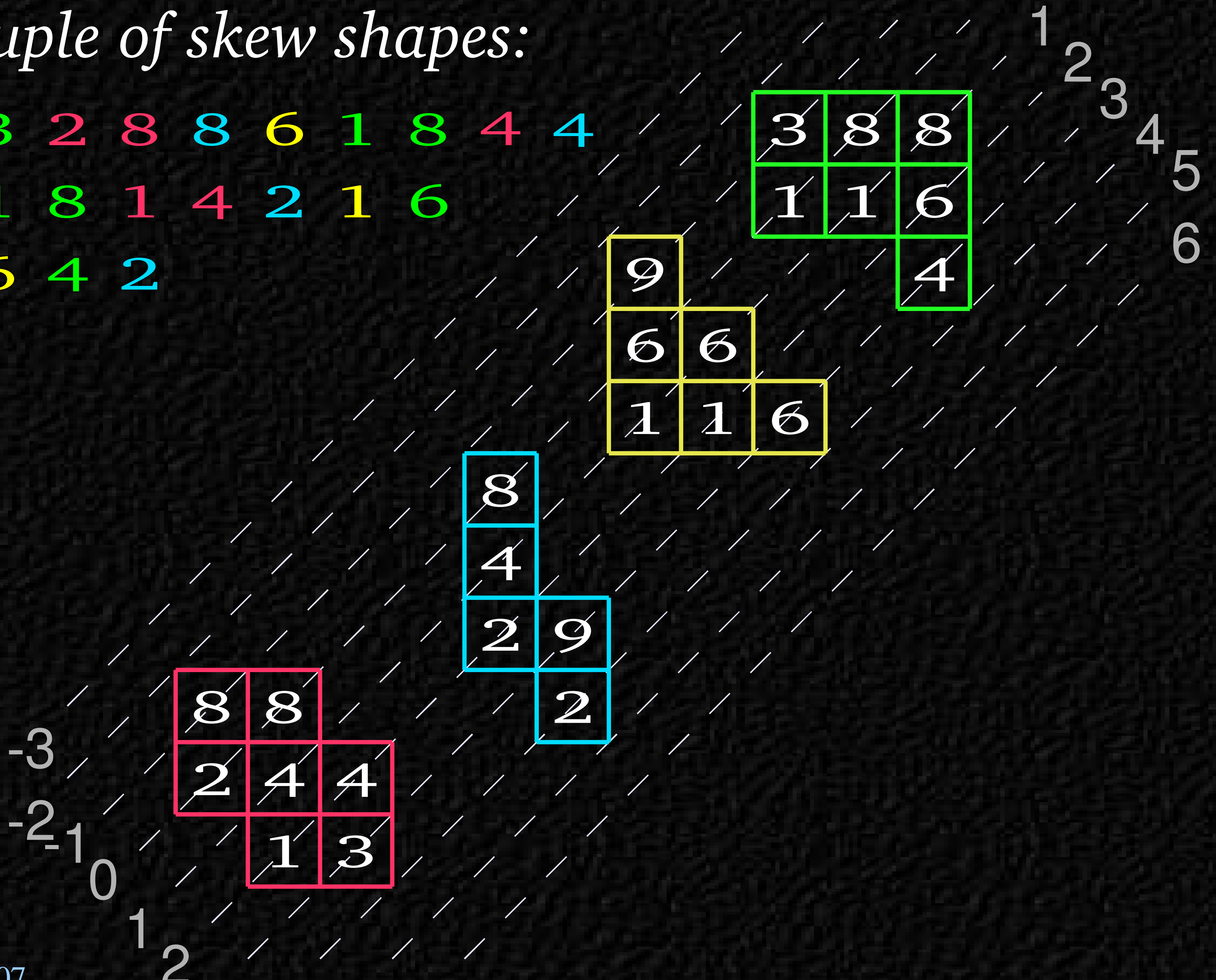


Aligning content lines in the k-quotient is equivalent to remembering the k-core — then we can use skew shapes.



Content reading word of a tableau on a tuple of skew shapes:

8 9 3 2 8 8 6 1 8 4 4
1 6 1 8 1 4 2 1 6
3 9 6 4 2



Inversions in the reading word count if they are on the ***same line***—

8 9 3 2 8 8 6 1 8 4 4

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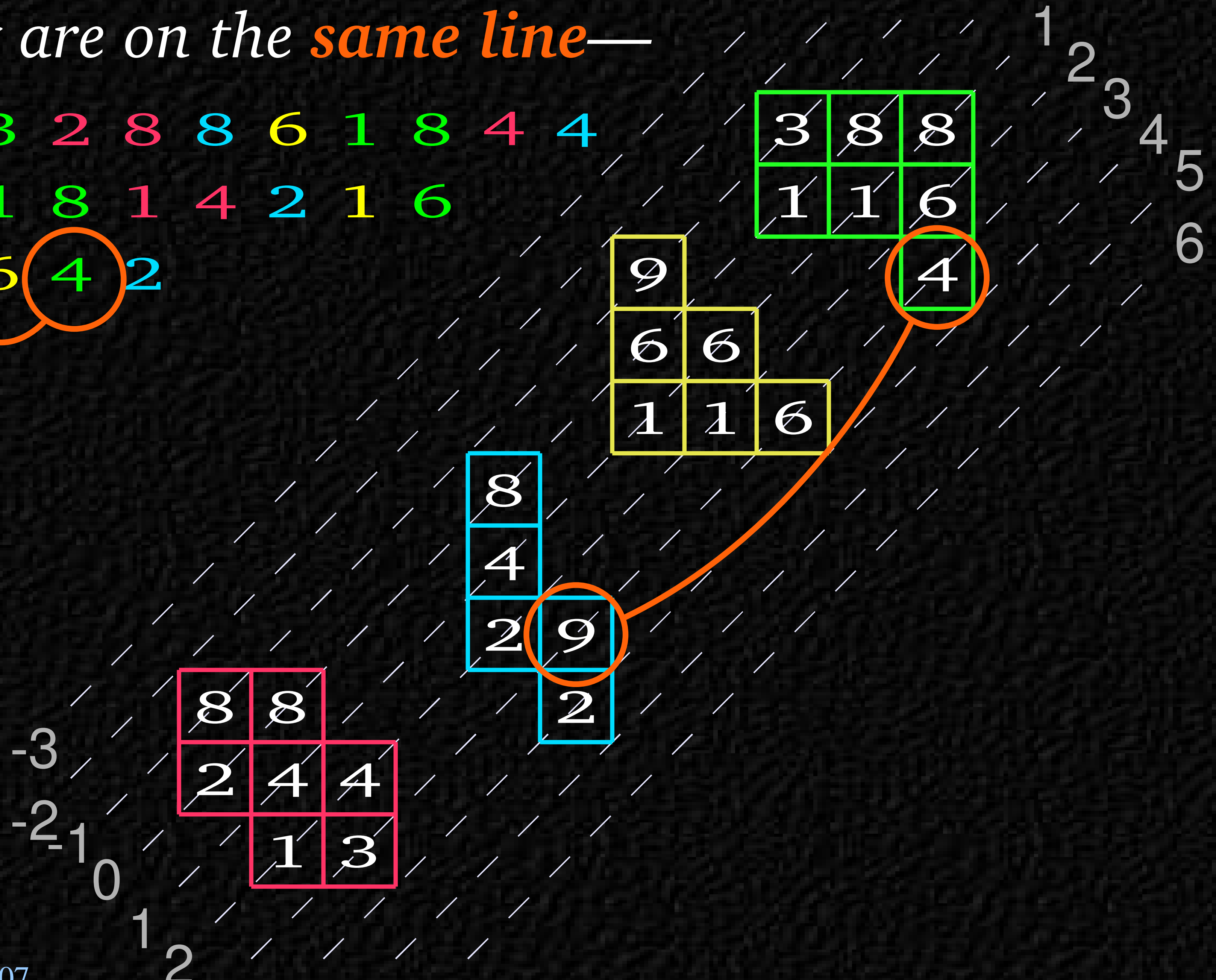

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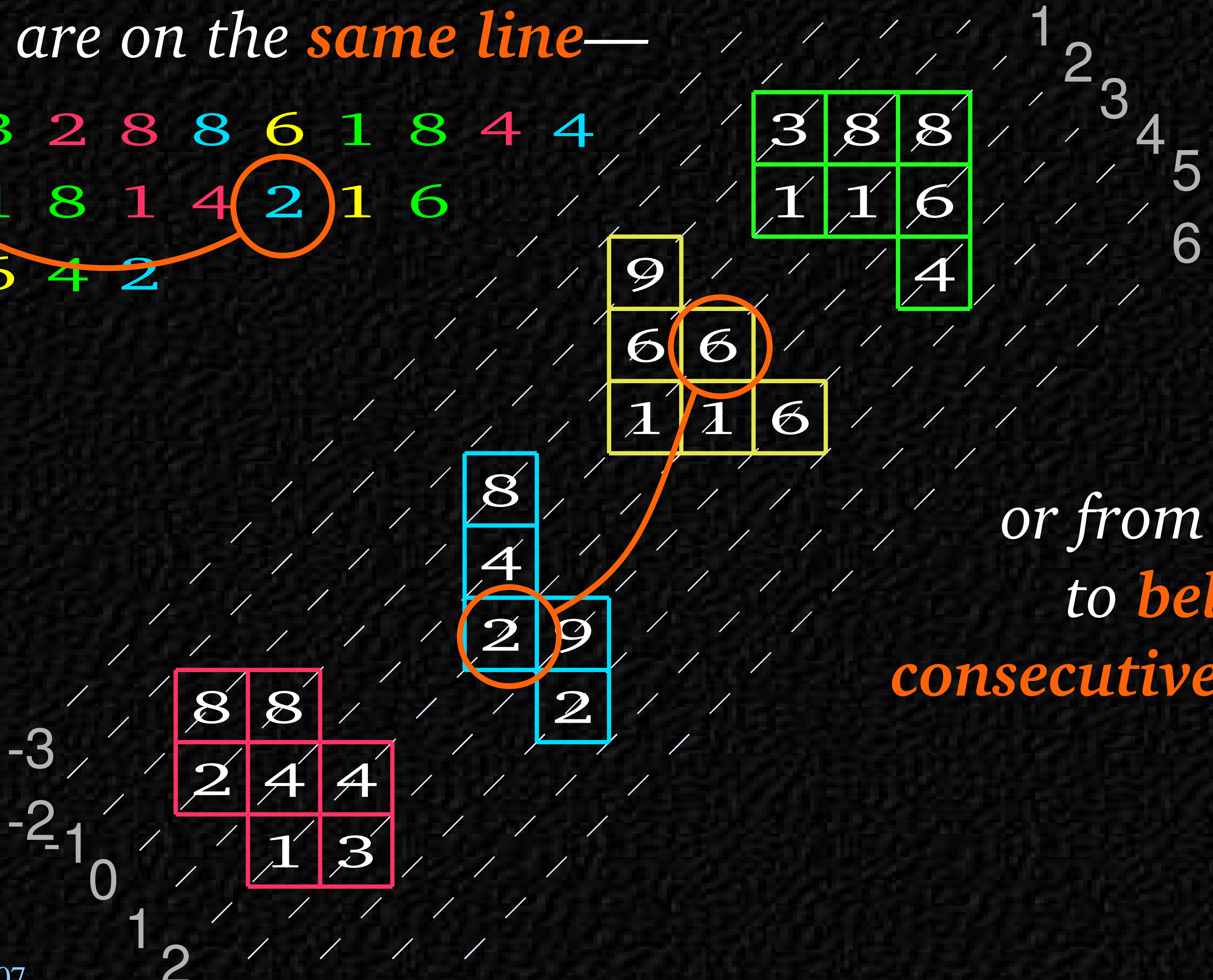

or from ***above*** *to* ***below*** *on* ***consecutive lines***.

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if they are on the same line—

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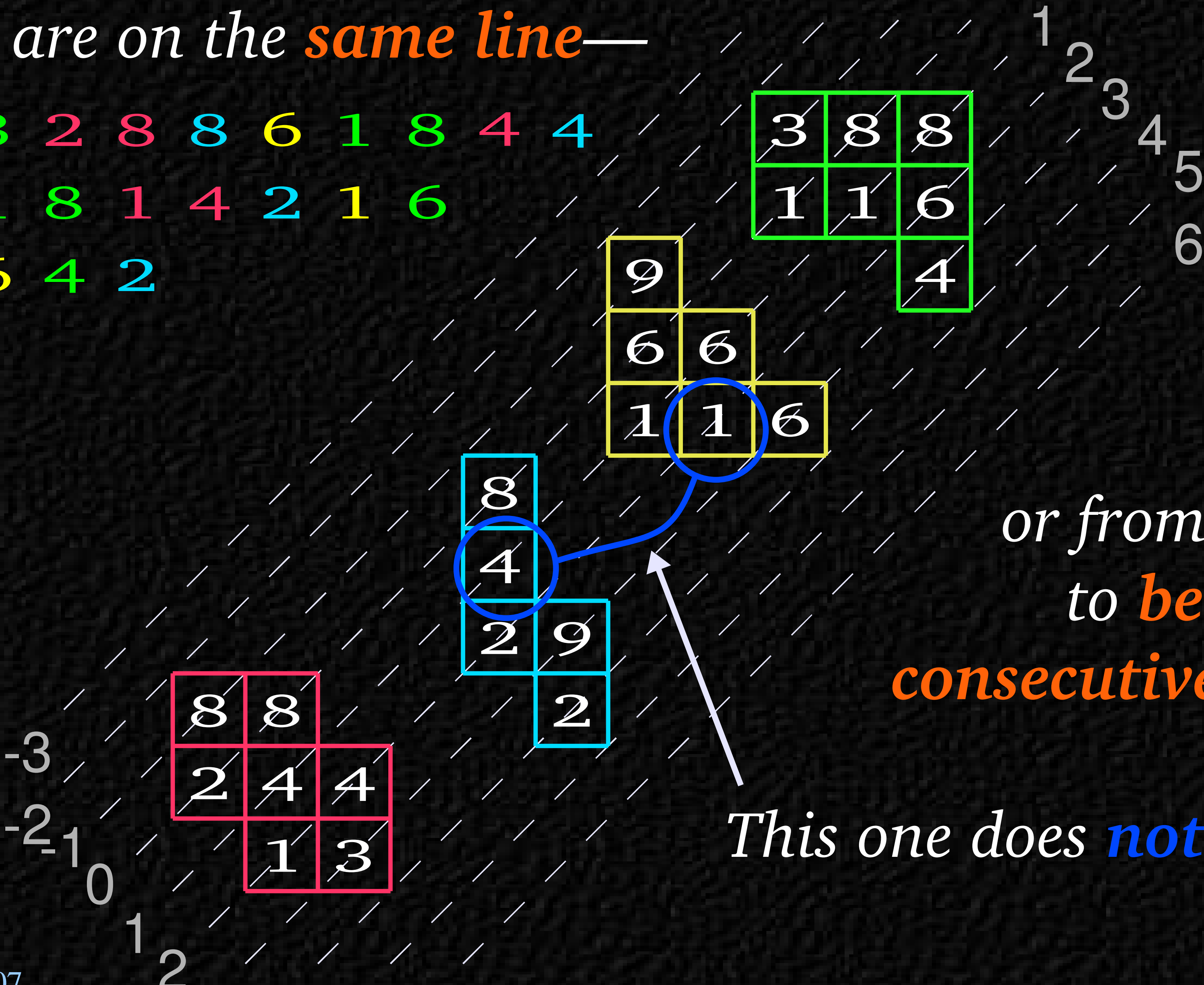

*or from **above** to **below** on **consecutive lines**.*

*This one does **not** count!*

Definition. *The* spin *of a ribbon tableau is the sum over its ribbons*

$$\mathrm{spin}(T) = \sum_R (\mathrm{height}(R) - 1)$$

Proposition. *If*

semistandard ribbon tableau T

$\longleftrightarrow$ *tuple of semistandard tableaux* S,

then for some constant e *depending only on the shape, we have*

$$\mathrm{spin}(T) = e - 2\,\mathrm{inv}(S)$$

Definition. *Given a tuple of skew shapes* $\boldsymbol{\nu}$ *(with content alignment), the associated* ***LLT polynomial*** *is the generating function for semistandard tableaux*

$$G_{\boldsymbol{\nu}}(x;q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} x^T$$

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Leclerc and Thibon conjectured that $G_{\nu}(x;q)$ is always *Schur positive*, and proved it for *straight* shapes. In this case, the coefficients in the Schur expansion are *parabolic Kazhdan-Lusztig polynomials*.

Haglund, H., and Loehr showed that the Macdonald polynomials $\tilde{H}_{\mu}(x;q,t)$ are positive combinations of LLT polynomials for certain *skew* shapes.

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We prove the Leclerc-Thibon conjecture for all shapes.

Part II *LLT Polynomials in the general case*

Notation:

$$G = \text{reductive algebraic group}$$
$$L = \text{Levi subgroup}$$
$$W = \text{Weyl group of } G$$
$$W_J = \text{Weyl group of } L \text{ (so } L = \bigcup_{w \in W_J} BwB)$$
$$W^J = \{\text{Minimal representatives of cosets } W_J\, w\}$$
$$X = \text{Weight lattice}$$

Hecke algebra $\mathcal{H}(W)$ *has basis* $\{T_w : w \in W\}$ *and relations*

$$T_v T_w = T_{vw} \quad \text{if} \quad l(vw) = l(v) + l(w)$$

$$(T_{s_i} - q)(T_{s_i} + 1) = 0$$

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Bernstein's presentation of the Affine Hecke algebra:

$$\mathcal{H}_{\text{aff}} = \mathcal{H}(W) \cdot \{Y^\lambda : \lambda \in X\}$$

$$T_{s_i} Y^\lambda - Y^{s_i(\lambda)} T_{s_i} = (q-1)\frac{Y^\lambda - Y^{s_i(\lambda)}}{1 - Y^{-\alpha_i}} \quad (i \neq 0)$$

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The center of the affine Hecke algebra is

$$Z(\mathcal{H}_{\text{aff}}) = (\mathbb{Q}(q) Y^X)^W$$

Submodule $\mathcal{H}(W)e^+$ *affords the "trivial" representation*

$$T_w e^+ = q^{l(w)} e^+$$

where $e^+ = \sum_{w \in W} T_w$.

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Theorem *(Lusztig). Let* w *be the maximal element of the double coset* $W\tau(\lambda)W \subset W_{\text{aff}}$. *The corresponding Kazhdan-Lusztig basis element in the affine Hecke algebra is*

$$C_w = \chi^\lambda(Y)e^+$$

where $\chi^\lambda(Y)$ *is an irreducible character of* G , *viewed as an element of the center* $Z(\mathcal{H}_{\text{aff}})$.

Submodule $e_J^- \mathcal{H}(W_J)$ *affords the sign representation*

$$e_J^- T_w = (-1)^{l(w)} e_J^- \quad \text{for} \quad w \in W_J$$

of $\mathcal{H}(W_J)$ *, where*

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The elements

$$|\lambda\rangle = e_J^- w_\lambda e_+$$

where $\lambda \in X_{++}(L)$ *is regular and dominant for* L *,*
and $w_\lambda \in W_J \tau(\lambda) W$ *is a minimal double coset representative in* W_{aff}*, form a basis of the space*

$$e_J^- \mathcal{H}_{\text{aff}}\, e^+$$

Note that $e_J^- \mathcal{H}_{\text{aff}} e^+$ *is a* $Z(\mathcal{H}_{\text{aff}})$*-module.*

The operator of multiplication by $\chi^\lambda(Y)$ *on the basis* $|\boldsymbol{\lambda}\rangle$ *has matrix entries denoted by*

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Definition. *The generating function of matrix entries*

$$\mathcal{L}^G_{L,\beta,\gamma}(x;q) = \sum_\lambda \langle\beta|\, \chi^\lambda(Y)\, |\gamma\rangle\, \chi^\lambda(x)$$

taken over all λ *, for fixed* β *and* γ*, is an* ***LLT polynomial***.

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[Remark: it's really an infinite formal q-character of G *.]*

Proposition. *Let* ρ_L *be such that* $\langle \alpha_j^\vee, \rho_L \rangle = 1$ *for all* $j \in J$. *Then, formally, at* $q = 1$ *we have*

$$\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(x;1) = \mathbf{Ind}^G_L(\chi^\beta_L \otimes (\chi^\gamma_L)^*)$$

In other words, the coefficient of $\chi^\lambda(x)$ *in* $\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(x;1)$ *is equal to the multiplicity of* χ^β_L *in* $\chi^\gamma_L \otimes \chi^\lambda_G|_L$.

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For $G = GL_n$, $L = GL_{m_1} \times \cdots \times GL_{m_k}$, *we get a product of skew Schur functions*

$$\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(x;1)_{\mathrm{pol}} = S_{\beta_1/\gamma_1} \cdots S_{\beta_k/\gamma_k}$$

where $\gamma = \gamma_1|\cdots|\gamma_k$ *and* $\beta = \beta_1|\cdots|\beta_k$.

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Warning: *when* $q \neq 1$, *the LLT polynomial*

$$\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(x;q)$$

depends on the choice of ρ_L *!*

Theorem. *For* $G = GL_n$, $L = GL_{m_1} \times \cdots \times GL_{m_k}$, $\gamma = \gamma_1 | \cdots | \gamma_k$ *and* $\beta = \beta_1 | \cdots | \beta_k$, *if we take* $\rho_L = \rho_{m_1} | \cdots | \rho_{m_k}$, *where* $\rho_m = (0, -1, \ldots, 1 - m)$, *then*

$$\mathcal{L}^G_{L, \beta + \rho_L, \gamma + \rho_L}(x; q)_{\mathrm{pol}} = q^? G_{(\beta_1/\gamma_1, \ldots, \beta_k/\gamma_k)}(x; 1/q)$$

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Corollary. *The coefficients in the expansion through Schur functions of the Macdonald polynomials* $\tilde{H}_\mu(x;q,t)$ *are positive.*

Part III *A few words about proofs—*

Two things to prove...

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We adapt the method of Leclerc and Thibon.

Rename the basis of $e_J^- \mathcal{H}_{\rm aff}\, e^+$, *denoting basis elements by* $|\underline{\mu}\rangle$, *where* μ *is a partition with fixed* k-*core and at most* n *parts. When* $|\underline{\mu}\rangle$ *is expressed using the Bernstein generators, its definition extends naturally to all* μ *in a* $W_{\rm aff}$-*orbit, with simple straightening relations.*

Two things to prove...

First, theorem *from previous slide: for* $G = GL_n$ *,*

$$\mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(x;q)_{\mathrm{pol}} = q^? G_{(\beta_1/\gamma_1,\ldots,\beta_k/\gamma_k)}(x;1/q)$$

It's enough to show that the operator $e_r(Y)$ *acts by adding a vertical ribbon strip* R *to* μ*, with coefficient*

$$q^{-\mathrm{spin}(T)/2}$$

This follows easily from the straightening relations.

Everything but the combinatorial action of $e_r(Y)$ *works for any* G *and* L *.*

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Problem. *Find a combinatorial formula for* $\mathcal{L}^G_{L,\beta,\gamma}(x;q)$ *for general* G *and* L.

Two things to prove...

Second,

Theorem *(Grojnowski, H.). The matrix coefficient*

$$\langle\beta|\,\chi^{\lambda}(Y)\,|\gamma\rangle$$

(the coefficient of $\chi^{\lambda}(x)$ *in* $\mathcal{L}^{G}_{L,\beta+\rho_L,\gamma+\rho_L}(x;q)$ *) is always positive, i.e., in* $\mathbb{N}\langle\langle q\rangle\rangle$ *.*

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which is a corollary to—

Theorem (G & H). Given any (possibly infinite) Weyl group W *and parabolic* $W_J \subseteq W$*, define* $TC_w = T_x C_y$ *, where* $w = xy$ *,* $x \in W^J$ *and* $y \in W_J$*. The matrix coefficients of a KL basis element* C_v *, acting by left multiplication on the basis* $\{TC_w\}$ *of* $\mathcal{H}(W)$ *, are positive.*

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Proof uses the standard picture of $\mathcal{H}(W)$ as convolution algebra of MHM´s or étale perverse sheaves (take your pick) on the flag variety…

…plus a maximally delicate variant of the usual reasoning [Springer, Lusztig], using purity of hyperbolic restriction [Braden, others].

The Moral

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...some are friendly!