On the H polynomials of reductive monoids

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What is a reductive monoid?

Naively speaking, it is the Zariski closure of a reductive group. More precisely, suppose $\rho : G_0 \to GL(V)$ is a (rat.) representation. Then

$$M(\rho) := \overline{\mathbb{C}^* \cdot \rho(G_0)} \subseteq End(V),$$

is a reductive monoid.

We will denote by G the group of invertible elements in $M(\rho)$.

Some questions about reductive monoids:

- The unit group G (hence any Borel subgroup) of the monoid acts on M. What can be said about the orbits?
- How do you classify them?
- What is the representation theory?

For the answers and other useful stuff see the text book: Linear Algebraic Monoids by Lex Renner. Also, there is an excellent *exposé* by Lois Solomon, called An introduction to reductive monoids.

The following two examples are from the Solomon's article.

Example 1. Let $V = \mathbb{C}^4 \otimes \mathbb{C}^4$, and consider $\rho : SL_4 \to GL(V)$ defined by

$$\rho(g)(v \otimes v') = gv \otimes gv'.$$

Then, $\mathbb{C}^* \cdot \rho(SL_4) = \{g \otimes g|\ g \in GL_4\}$ and hence

$$\begin{aligned} M(\rho) = \overline{\mathbb{C}^* \cdot \rho(SL_4)} &= \{a \otimes a|\ a \in M_4\} \\ &\cong M_4. \end{aligned}$$

Example 2. Now, consider $\sigma : SL_4 \to GL(V)$ defined by

$$\sigma(g)(v \otimes v') = gv \otimes (g^{-1})^t v'.$$

Then the unit group of $M(\sigma)$ is very similar to that of $M(\rho)$, however, these monoids are different in a fundamental way.

The difference can be read off from the idempotents $E(M(\rho)) = \{e \in M(\rho) | e^2 = e\}$.

$T \subseteq G$ is a maximal torus, then $M(\rho)$ contains the affine toric variety $\overline{T}$. Therefore $E(\overline{T}) \subseteq E(M(\rho))$.

$E(M(\rho))$ is a poset: $e \leq f \iff e = fe$. We consider $E(\overline{T})$ with the induced partial order.

Theorem -an eye opener: Let $T \subseteq \mathbf{T_n}$ be a subtorus of the diagonal invertible $n \times n$ matrices. Let $\chi_1, ..., \chi_n \in X(T)$ be the restrictions of the coordinate functions on $\mathbf{T_n}$ to T. Let

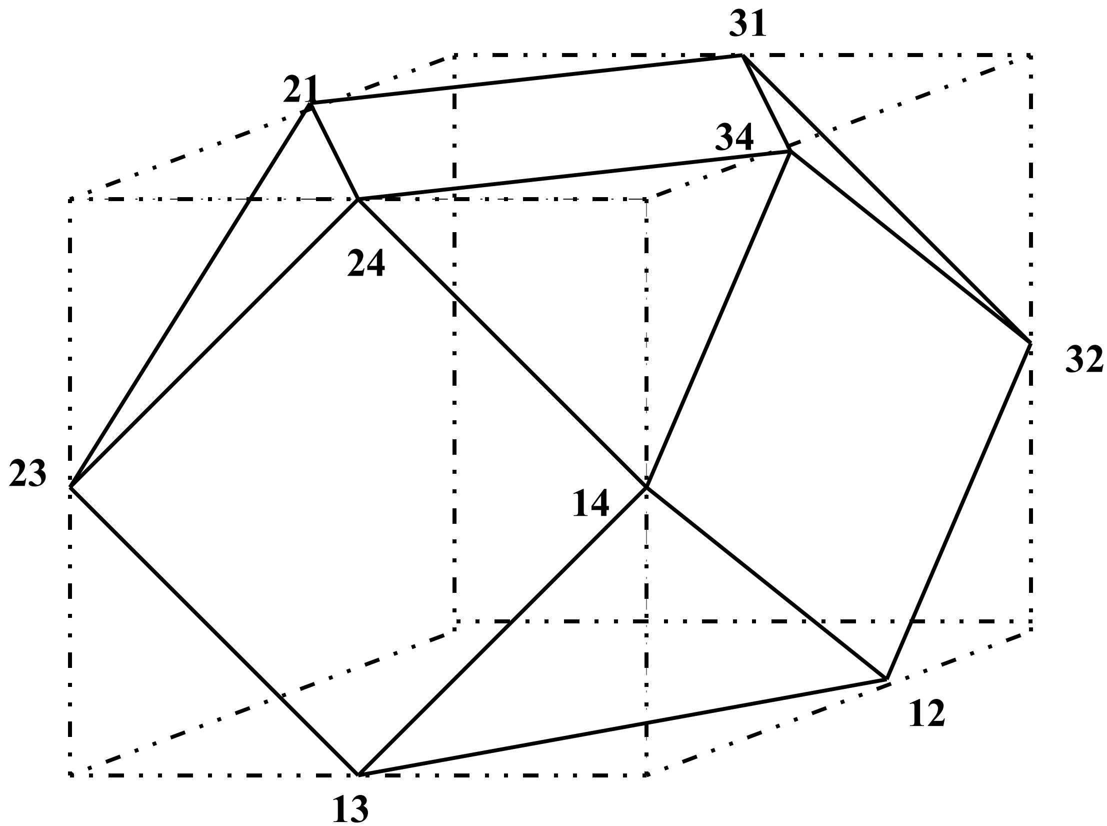
$$\mathcal{L} = \{\lambda \in X_* | \ \langle \chi_i, \lambda \rangle \geq 0, \text{for } 1 \leq i \leq n\}$$

be the associated polyhedral cone.

Then the face lattice of $\mathcal{L}$ is anti-isomorphic to the lattice of idempotents $E(\overline{T})$.

If furthermore G is semisimple and $0 \in M$, then we can replace the face lattice of the cone with the face lattice of a polytope.

In this spirit;
- $E(\overline{T}) \subseteq M(\rho)$ of the example 1 is isomorphic to the face lattice of the standard 4-simplex.
- $E(\overline{T}) \subseteq M(\sigma)$ is the face lattice of the cuboctahedron.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Definition. The cross section lattice $\Lambda \subseteq E(\overline{T})$ is the sublattice

$$\Lambda := \{e \in E(\overline{T}) \mid Be \subseteq eB\}.$$

Theorem (Putcha) Let M be a reductive monoid with the unit group G. Then

$$M = \bigsqcup_{e \in \Lambda} GeG$$

Let $R = \overline{N_G(T)}/T$, where $\overline{N_G(T)}$ is the Zariski closure in M.

Theorem (Renner)

- R is a finite monoid.
- The group of units of R is the Weyl group W, and $R = WE(R)$,
- $E(R) = E(\overline{T})$,
- For $e \in \Lambda$, $GeG = \bigsqcup_{r \in WeW} BrB$,
- $M = \bigsqcup_{r \in R} BrB$,
- Bruhat-Chevalley order on W extends to R.

Remark. For $r \in R$,

$$BrB \cong \mathbb{C}^{\ell(r)-rk(r)} \times (\mathbb{C}^*)^{rk(r)},$$

where $rk(r) = dim(Tr)$ and $\ell(r) = dim(BrB)$.

Definition. (Renner)

$$H(M,q) = \sum_{r \in R} q^{\ell(r)-rk(r)}(q-1)^{rk(r)}.$$

Remarks.

• This definition works for any variety with finitely many $B \times B$ orbit.

• $H(M, q) = \sum_{e \in \Lambda} H(GeG, q)$.

• If M is *quasi-smooth*, then Renner shows that $(H(M, q) - 1)/(q - 1)$ is the intersection homology Poincare polynomial of $M \backslash \{0\}/\mathbb{C}^*$.

Question. Would it be interesting to study

$$H_M(q, t) = \sum_{r \in R} q^{\ell(r) - rk(r)} t^{rk(r)}$$

Answer: Of course!

Theorem (Can, Renner) Let M be a reductive monoid, and let $e \in \Lambda$. Then there exists a $B \times B$ equivariant fibration

$$\mathcal{G}(e) \to GeG \to G/P \times G/P^{-},$$

where P is a maximal parabolic subgroup and $\mathcal{G}(e)$ is a unit group of a submonoid of M.

Corollary: $H_{GeG}(q,t) = H_{G/P}(q,t)^2 H_{\mathcal{G}(e)}(q,t)$.

Theorem.(C., R.) Let $M = M_n$ be the monoid of $n \times n$ matrices. Then, the H-polynomial $H_M(q,t)$ is equal to

$$H_M(q,t) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{\binom{k}{2}} t^k.$$

Something hilarious: replace t by $q-1$, then everything cancels to q^{n^2}.

Classic Laguere polynomials:

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k x^k}{(\alpha+1)_k k!},$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$, and $\alpha \in \mathbb{C}$. These polynomials satisfy the orthogonality relation

$$\int_0^\infty L_n^{(\alpha)}(x) L_m^{\alpha}(x) x^{(\alpha)} e^{-x} dx = \delta_{mn} \Gamma(\alpha+n+1)/n!,$$

Moak's q-analogue of the Laguere polynomials is defined as

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n} \frac{(q^{-n};q)_k q^{\binom{k}{2}}(1-q)^k(q^{n+\alpha+1}x)^k}{(q^{\alpha+1};q)_k(q;q)_k},$$

where $(q^a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$. These also satisfy certain orthogonality relations similar to classical case.

Theorem.(C., R.)

$$H_{M_n}(q,t) = t^n q^{-\binom{n}{2}}[n]_q! L_n^{(0)}(\frac{-1}{tq^{n-1}}; q)$$

Corollary.
The length generating function $\sum_{r \in R_{M_n}} q^{\ell(r)}$ is given by

$$H_{M_n}(q,q) = q^{n-\binom{n}{2}}[n]_q! L_n^{(0)}(-q^n; q)$$

q-**Rook polynomials**

$$R_k(\mathcal{F}; q) = \sum_C q^{inv(C,\mathcal{F})}$$

Here $\mathcal{F}$ is a right justified *Ferrers board* in an $n \times n$ grid of squares, and the sum is over all placements C of k *nonattacking rooks* on the squares of $\mathcal{F}$

	x			x	x	
0	x			x	x	
0	*	x	x	x	x	x
0	0	0		*	x	x
0	0	0			x	
0	0	0			x	
0	0	0	0	0	*	x

Theorem. (C., R.) Let $\mathcal{F}$ be a Ferrers board of shape λ. And let $M^k_{\lambda,n} \subseteq M_n$ be the set of all rank k matrices of shape λ. Then the $(q,t)-H$ polynomial of $M^k_{\lambda,n}$ is given by

$$H_{M^k_{\lambda,n}}(q,t) = t^k q^{|\lambda|-k} R_k(\mathcal{F}; \frac{1}{q}).$$

Note: $\cup_{k=0}^n M^k_{\lambda,n}$ is an affine subspace of M_n of dimension $|\lambda|$.

Definition. Let $Sp_n = \{g \in GL_n | g^t J g = J\}$ be the symplectic group, where $n = 2l$, $J = \begin{pmatrix} 0 & E_l \\ -E_l & 0 \end{pmatrix} \in M_n$, where E_l is the $l \times l$ anti-diagonal $(1, ..., 1)$. Set $G = \mathbb{C}^* \cdot G_0 \subseteq GL_n$. Then the symplectic monoid MSp_n is defined as the Zariski closure of G in M_n.

Theorem.(C., R.) The (q,t)-H polynomial of the symplectic monoid MSp_n is

$$H_{MSp_n}(q,t)$$

$$= 1+\sum_{k=1}^{l} q^{(l-k)^2} t^{l-k+1} \frac{[2l]!!^2}{[2l-2k]!![k]!^2} + q^{l^2} t^{l+1} [2l]_q!!$$

We have been thinking about/work in progress:

• We can define H polynomials for matrix Schubert varieties. In fact, we can do it for any interval in the poset R (w.r.t. Bruhat-Chevalley order). So, we have been thinking about the relationship between intersection homology Poincare polynomial and the H-polynomials.

• A conjecture of Garsia and Remmel says that q-Rook polynomials are unimodal for any λ and k. Remember Stanley's proof of unimodality for h-polynomials. So, we have been thinking about applying Hard-Lefchetz theorem..

• Other families of orthogonal polynomials specializing to the (q,t)-H polynomial of MSp_n or of MSO_n.