



$$\langle \mathcal{O}_\mu, \mathcal{O}_\nu^* \rangle_e = \langle \mathcal{O}_\mu, \mathcal{O}_\nu^*, 1 \rangle_e$$

$$QK(X) = K(X) \otimes \mathbb{Z}[[q]]$$

$$\mathcal{O}_\lambda * \mathcal{O}_\mu = \sum_{\nu, d \geq 0} N_{\lambda, \mu}^{\nu, d} q^d \mathcal{O}_\nu$$

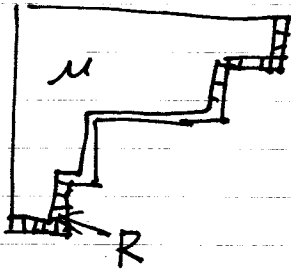
Thm (Givental) Associative!

$$\text{Thm (B-M)} \quad \mathcal{O}_i \cdot \mathcal{O}_\mu = \sum_{\text{letters}} C_{i, \mu}^\lambda \mathcal{O}_\lambda + \sum_{\nu} N_{i, \mu}^{\nu, 1} q \cdot \mathcal{O}_\nu$$

$$N_{i, \mu}^{\nu, d} = 0 \text{ for } d \geq 2.$$

$$N_{\lambda, \mu}^{\nu, 1} \neq 0 \implies \ell(\mu) = m$$

and ν is obtained from μ by removing boxes from outer rim R of μ ≥ 1 box from each row.



$$\implies N_{\lambda, \mu}^{\nu, 1} = (-1)^e \binom{s}{e}$$

$$e = |\nu| + n - i - |\mu|, \quad s = \# \text{ rows of } \nu \cap (R \setminus \text{last row}).$$

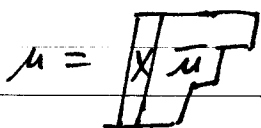
Example $X = Gr(3, \mathbb{C}^6)$

$$N_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}, \begin{smallmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{smallmatrix}}^{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}, 1} = -2$$

$$\mathcal{O}_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} \cdot \mathcal{O}_{\begin{smallmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{smallmatrix}} = \mathcal{O}_{\begin{smallmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{smallmatrix}} + q \mathcal{O}_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} + q \mathcal{O}_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} - q \mathcal{O}_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}}.$$

$$s=2, e=1$$

Grambelli: Thm $\mathcal{O}_{\begin{smallmatrix} \square \\ \mu \end{smallmatrix}} = \sum_{\substack{p \geq a \\ \bar{\mu} \subset \nu \subset \mu}} (-1)^{|\mu|} \binom{p-a-i+c(\nu/\bar{\mu})}{p-a-|\mu|} \mathcal{O}_p * \mathcal{O}_\nu$



\uparrow
 $QK(X)$.