Diagonal Invariants of the symmetric group and products of linear forms

CRM, May 29, 2007

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Products of linear forms= "Decomposable forms"

$\mathbb{K}$: algebraically closed field.

"Decomposable forms" of degree n in $\mathbb{K}[t_0, t_1, \ldots, t_r]$

$=$

totally factorizable forms

$=$

products of n linear forms.

They are a closed algebraic subvariety in the space of forms of degree n. What is its ideal ?

Two nice works on the problem

John Dalbec, *Multisymmetric functions*, Beiträge zur Algebra und Geometrie 40 (1999).

Friedrich Junker, *Über symmetrische Functionen von mehreren Reihen vor Veränderlichen*, Matematische Annalen 43 (1893).

Google translation

(up to permutation)

Defining maps for the subvariety of decomposable forms

V : $\mathbb{K}$–vector space of dimension $r+1$.

The subvariety of decomposable forms is the image of:

$$\begin{array}{rccc} \pi : & (V^*)^n/\mathfrak{S}_n & \longrightarrow & S^n V^* \\ & (\ell_1, \ell_2, \dots, \ell_n) \ \text{mod}\ \mathfrak{S}_n & \longmapsto & \ell_1 \cdot \ell_2 \cdots \ell_n \end{array}$$

It is the affine cone over the image of:

$$\begin{array}{rccc} \mathbb{P}\pi : & \mathit{Symm}^n\,\mathbb{P}^r = (\mathbb{P}V^*)^n/\mathfrak{S}_n & \longrightarrow & \mathbb{P}(S^n V^*) \\ & (H_1, H_2, \dots, H_n) \ \text{mod}\ \mathfrak{S}_n & \longmapsto & H_1 + H_2 + \cdots + H_n \end{array}$$

The map $\mathbb{P}\pi$ is injective and its image, $\mathit{Chow}(n, 0, \mathbb{P}^r)$, is the Chow variety of the 0-cycles of degree n in $\mathbb{P}^r$.

Associated maps of graded algebra

$S^n V =$ symmetric power over V (quotient of $\otimes^n V$).

$T^n_{sym} V =$ symmetric tensors over V (subspace of $\otimes^n V$).

The map

$$\mathbb{P}\pi : \mathit{Symm}^n\,\mathbb{P}^r = (\mathbb{P}V^*)^n/\mathfrak{S}_n \quad \longrightarrow \quad \mathbb{P}(S^n V^*)$$

gives rise to a map between the homogeneous coordinate rings:

$$\pi^* : \bigoplus_{d=0}^{\infty} T^n_{sym} S^d V \quad \longleftarrow \quad \bigoplus_{d=0}^{\infty} S^d T^n_{sym} V = S^\bullet T^n_{sym} V$$

Questions

$$\begin{array}{rccc} \mathbb{P}\pi : & \mathit{Symm}^n\,\mathbb{P}^r & \longrightarrow & \mathbb{P}(S^nV^*) \\ & (H_1, H_2, \ldots, H_n) \ \text{mod}\ \mathfrak{S}_n & \longmapsto & H_1 + H_2 + \cdots + H_n \end{array}$$

Pb 1. Is $\mathbb{P}\pi$ an isomorphism $\mathit{Symm}^n\,\mathbb{P}^r \cong \mathit{Chow}(n, 0, \mathbb{P}^r)$?

Pb 2. Compute, or describe, $\mathit{ker}\,\pi^*$ (defining equations for the subvariety of decomposable forms)

In coordinates

The map:
$\mathbb{P}\pi$: $\mathit{Symm}^n\,\mathbb{P}^r = (\mathbb{P}V^*)^n/\mathfrak{S}_n \quad\longrightarrow\quad \mathbb{P}(S^nV^*)$
can be worked out in homogeneous coordinates.

Write $\ell_i = a_{i0}t_0 + a_{i1}t_1 + \cdots + a_{ir}t_r$ and $\prod_i \ell_i = \sum_\alpha \hat{e}_\alpha\, t_0^{\alpha_0}t_1^{\alpha_1}\cdots t_r^{\alpha_r}$.

It appears as: $\hat{A} = \begin{bmatrix} a_{10} & a_{11} & \cdots & a_{1r} \\ a_{20} & \cdots & & \vdots \\ \vdots & & & \vdots \\ a_{n0} & \cdots & & a_{nr} \end{bmatrix} \mapsto \left(\hat{e}_\alpha(\hat{A})\right)_{|\alpha|=n}$

The algebra $\bigoplus_d T^n_{\mathit{sym}}SV$ of $\mathit{Symm}^n P^r$ is $\mathit{HDSym}^{r+1}_n(\mathbb{K})$ (*homogeneous diagonal invariants of the symmetric group*): the polynomials in the entries of $\hat{A}$, invariants under row permutations, homogeneous in the variables of each row.

The $\hat{e}_\alpha$ (*fundamental homogeneous invariants*) are a linear basis for the piece of degree 1 of $\mathit{HDSym}^{r+1}_n(\mathbb{K})$.

In coordinates, locally

The map:
$\mathbb{P}\pi$: *Symm*$^n\,\mathbb{P}^r = (\mathbb{P}V^*)^n/\mathfrak{S}_n \quad \longrightarrow \quad \mathbb{P}(S^nV^*)$
can be worked out in affine charts: $a_{i0} = 1$ for all i and $\widehat{e}_{n00\cdots0} = 1$.

Write $\ell_i = 1 + a_{i1}t_1 + \cdots + a_{ir}t_r$ (we set $t_0 = 1$)
and $\prod_i \ell_i = 1 + \sum_\alpha e_\alpha\, t_1^{\alpha_1} \cdots t_r^{\alpha_r}$ $(1 \leq |\alpha| \leq n\;)$.

It appears as: $\pi_{aff} : A = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & & a_{nr} \end{bmatrix} \mapsto (e_\alpha(A))_{1\leq|\alpha|\leq n}$

The algebra of the affine chart of *Symm*$^n P^r$ is *DSym*$_n^r(\mathbb{K})$ (*diagonal invariants of the symmetric group*): the polynomials in the entries of A, invariants under row permutations.

The e_α are the *elementary polynomials*. The algebra *DSym*$_n^r$ also contains analogues of the classical *power sums* and *monomial functions* and conversion algorithms.

The isomorphism problem

Pb 1. Is $\mathbb{P}\pi$ an isomorphism $Symm^n\,\mathbb{P}^r \cong Chow(n,0,\mathbb{P}^r)$?

Neeman (1989): if $char\,\mathbb{K} = 0$ or $> n$ then $\mathbb{P}\pi$ is an isomorphism.

Dalbec (1999): $\mathbb{P}\pi : Symm^2\,\mathbb{P}^2 \longrightarrow Chow(2,\mathbb{P}^2)$ is an isomorphism regardless of $char\,\mathbb{K}$.

E.B. (2002)

- For $char\,\mathbb{K} = 2$, $\mathbb{P}\pi$ is an isomorphism iff $r = 1$ or $n = 1$ or $r = 2$ with $n \leq 3$.

- For $char\,\mathbb{K} > 2$, $\mathbb{P}\pi$ is an isomorphism iff $n > char\,\mathbb{K}$ or $r = 1$.

The isomorphism problem

Idea of the proof:

This can be established locally.
$\mathbb{P}\pi$ is an isomorphism
iff π_{aff} (the map between affine charts) is an embedding
iff the e_α generate $DSym_n^r(\mathbb{K})$.

- Use a finite generating set of $DSym_n^r$ (over the integers) like Fleischmann's monomial functions or the Bergeron–Lamontagne basis.

- Use projections $DSym_n^r \to DSym_{n-1}^r$ and $DSym_n^r \to DSym_n^{r-1}$.

- Finish with a small number of small "brute force" computations.

From now on $\mathbb{K} = \mathbb{C}$

Pb. 2: Compute the ideal of Chow$(n, 0, \mathbb{P}^r)$

It is $\mathit{ker}\,\pi^* =$ ideal of the algebraic relations between the $\hat{e}_\alpha \in \mathit{HDSym}_n^{r+1}(\mathbb{C})$

It is easier to get $\mathit{ker}\,\pi^*_{\mathit{aff}} =$ ideal of the algebraic relations between the $e_\alpha \in \mathit{DSym}_n^r(\mathbb{C})$ because:

- A nice algorithm to produce all relations in given multidegree is known (Junker + Dalbec).

- The (multigraded) Hilbert series of DSym_n^r is known (it is the generating function for *vector partitions*). (Gessel–Garsia 1979, Bergeron–Lamontagne 2005)

All is in favour of *Hilbert–driven Gröbner basis computations*.

Compute the ideal of Chow$(n, 0, \mathbb{P}^r)$

Two types of monomial orderings are interesting:

1. total degree order.
Useful: A Gröbner basis for $ker\,\pi^*_{aff}$ (relations between elementary polynomials) provides a Gröbner basis for $ker\,\pi^*$ by mere homogenization of its elements.

2. "easy" order, enjoying the structure of $DSym^r_n$.
$DSym^r_n$ is a free module of rank $(n!)^{r-1}$ over a subalgebra $\cong \otimes^r Sym_n$.
This provides small Gröbner bases and smaller (non Gröbner) generating sets.

small set of generators / # gb for nice order / # gb for total degree

$Symm^n\,\mathbb{P}^r$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$r = 2$	1/1/1	5/5/35	15/23/1139	35/102/?	70/518/?
$r = 3$	6/6/12	43/53/1779	177/743/?		
$r = 4$	20/20/	196/292/?			

Dalbec (1999) conjectured that the ideal of $Chow(3, 0, \mathbb{P}^3)$ is generated in degree 4: true.

Foulkes–Howe conjecture

Foulkes' (open) plethysm conjecture: (1950)

$h_d \circ h_n - h_n \circ h_d$ is Schur–positive for $d \geq n$.

Consider

$$\pi^* = \oplus_d \pi_d^* \; : \; \bigoplus_{d=0}^{\infty} T_{sym}^n S^d V \longleftarrow \bigoplus_{d=0}^{\infty} S^d T_{sym}^n V$$

$GL(V)$–characters for the pieces of degree d:

$h_n \circ h_d$ and $h_d \circ h_n$

Foulkes–Howe conjecture

Foulkes' (open) plethysm conjecture: (1950)

$h_d \circ h_n - h_n \circ h_d$ is Schur–positive for all $d \geq n$.

Consider

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$GL(V)$–characters for the pieces of degree d:

$$h_n \circ h_d \qquad \text{and} \qquad h_d \circ h_n$$

Howe's (stronger) conjecture(s) (1987)

FH (i) π_d^* is injective for all $d \leq n$.

FH (ii) π_d^* is surjective for all $d \geq n$.

Foulkes–Howe conjecture

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$GL(V)$–characters for the pieces of degree d:

$$h_n \circ h_d \qquad \text{and} \qquad h_d \circ h_n$$

Howe's (stronger) conjecture(s) (1987):

FH (i) π_d^* is injective for all $d \leq n$.
($\Leftrightarrow$ π_n^* is bijective)
($\Leftrightarrow$ No form of degree $\leq n$ vanishes on $Chow(n, 0, \mathbb{P}^r)$) ($\Leftrightarrow$ The degree n piece of $HDSym_n^{r+1}$ is generated by the fundamental homogeneous invariants $\widehat{e}_\alpha$.)
($\Leftrightarrow$ $Chow(n, 0, \mathbb{P}^r)$ has no equation of degree $\leq n$)

Foulkes–Howe conjecture

M. Brion (1997): FH conjecture (ii) is true for $d >> n$ (with explicit lower bound depending on n and r)

E.B. : FH conjectures (i) and (ii) are true for $n = 3$.

E.B. (2002), J. Jacob (2004): FH conjecture (i) is true for $n = 4$.

Foulkes–Howe conjecture

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Müller+Neunhöffer (2005): FH conjectures (i) and (ii) are false for $n = 5$.

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Anyway . . . when is π_d^* injective ? surjective ?

Showing that FH (i) holds for fixed n: Toy example $n = 2$

To show: that the fundamental homogeneous invariants generate the degree n piece of *HDSym*$_n^{r+1}(\mathbb{C})$.

Ex: $n = 2$, $\qquad A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

The monomial functions = orbit sums of monomials $\Sigma(a_1^{\alpha_1} a_2^{\alpha_2} b_1^{\beta_1} b_2^{\beta_2})$ (under row permutations of the matrix) are a linear basis for *DSym*.

The decomposition

$$\Sigma(a_1^2 b_1 b_2) = e_{11} e_{20}$$

is obtained by applying the *polarization operator*

$$\tfrac{1}{2}\left(a_1 \frac{\partial}{d\,a_2} + b_1 \frac{\partial}{d\,b_2}\right)$$

to the *key identity*:

$$\Sigma(a_1^2 b_2^2) = a_1^2 b_2^2 + a_2^2 b_1^2 = \quad e_{11}^2 - 2e_{20}e_{02}$$

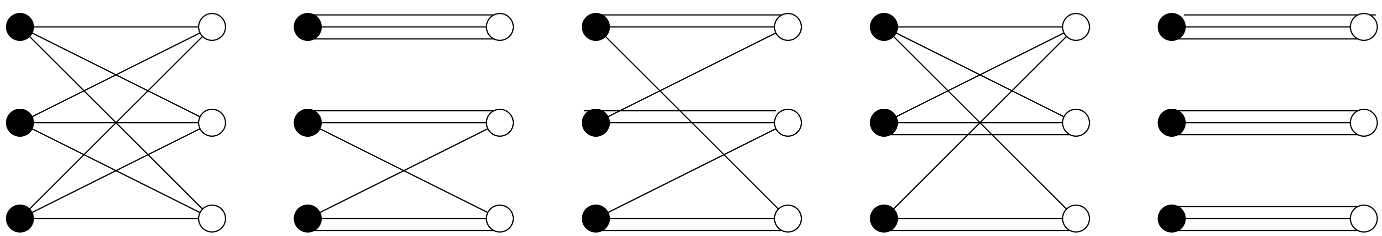
The key identity is also invariant under Column permutations !

Doubly symmetric functions

Checking FH (i) for fixed n can be reduced to computations in the subspace of $\mathbb{C}[A]^{\mathfrak{S}_n \times \mathfrak{S}_n}$: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$ of the elements homogeneous of degree n w.r.t. the variables of each row and homogeneous of degree n w.r.t. the variables of each column.

Ex: $n = 3$, linear bases are indexed by:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$



Even the enumeration of these objects is difficult !

Conference: Diagonally symmetric polynomials and applications.

October 15-19, 2007

Castro-Urdiales, Spain.

http://www.congreso.us.es/dsym/

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