Basic classes and embedded spheres

Wojciech Wieczorek

May 23, 2002

ABSTRACT: For a given smooth four manifold we study a relation between the simple type condition for Donaldson invariants and the intersections of basic classes with smoothly embedded spheres with self-intersection -2 and -3.
Studying the structure of Donaldson invariants imposed by existence of embedded spheres we also recover one criterion for manifold being of simple type (previously proven by Kronheimer–Mrówka using singular connections).
Keywords: Donaldson invariants, basic classes, embedded spheres.
AMS classification: 57,55.

1 Introduction

Every 2 dimensional cohomology class in a 4 dimensional smooth, simply connected manifold can be represented by smoothly embedded surface, but it can not always be represented by an embedded sphere. This raises the question of finding lower bounds on the genus of such surfaces. When X is a complex algebraic surface and S is an embedded complex algebraic curve of genus g, then the adjunction formula (see for example [5]), states that:

$$2g - 2 = K \cdot S + S \cdot S$$

where K is the canonical class of X. Within the framework of Donaldson theory, Kronheimer and Mrówka have introduced the notion of *basic classes*. These classes can be thought of as generalizations of the canonical class, since in a similar fashion they give restrictions on the genus of embedded surfaces in four manifolds. Later, the same restrictions, as well as many new ones, were formulated by using Seiberg-Witten invariants.

In Donaldson theory there is the crucial relation:

$$D_X(x^2) = 4D_X$$

that is satisfied by all known smooth manifolds with $b_+ > 1$. Manifolds X that satisfy this relation are said to be of *simple type*. For manifolds with this property, different Donaldson invariants can be combined into a formal power series:

$$\mathbb{D}(\alpha) = D\left(\left(1 + \frac{x}{2}\right)\exp(\alpha)\right)$$

This invariant $\mathbb{D}$ has a very nice structure:

Theorem 1.1 (Kronheimer, Mrówka [6]) *For a simply connected 4–manifold X of simple type there are finitely many* basic *classes $K_1, \ldots, K_s \in \mathrm{H}^2(X, \mathbb{Z})$ such that:*

1. $K_i \equiv w_2(X) \pmod 2$.
2. *There are rational numbers $a_1, \ldots, a_s$ such that*

$$\mathbb{D}_X = \exp(Q_X/2) \sum_{i=1}^{s} a_i \sinh K_i \qquad \text{if } b_+ \equiv 1 \pmod 4\text{, or} \tag{1.2}$$

$$\mathbb{D}_X = \exp(Q_X/2) \sum_{i=1}^{s} a_i \cosh K_i \qquad \text{if } b_+ \equiv 3 \pmod 4. \tag{1.3}$$

where Q_X denotes thee intersection form on X.

The importance of these basic classes lies in the following generalization of the adjunction formula, which is referred to as the *adjunction inequality*:

Theorem 1.4 (Kronheimer, Mrówka [8]) *If X is a simply connected $4-$manifold of simple type with basic classes $K_1, \ldots, K_s$, and S is any smoothly embedded, essential connected surface in X, such that $S \cdot S > 0$, then the genus g of S satisfies the lower bound:*

$$2g - 2 \geq K_i \cdot S + S \cdot S$$

for $i = 1, 2, \ldots, s$.

In 1994 Witten discovered that certain monopole equations define new invariant of four manifolds

$$SW_X : \mathrm{H}^2(X) \to \mathbb{Z}$$

Introducing these invariants he conjectured that the basic classes of Donaldson theory correspond to the characteristic classes $L \in \mathrm{H}^2(X)$ such that $SW_X(L) \neq 0$. In the same paper Witten also conjectured that the simple type condition in Donaldson theory is equivalent to the condition that for every characteristic class L such that the dimension of moduli space $\mathcal{M}_X(L)$ is not zero the invariant $SW_X(L) = 0$.

Soon after the discovery of Seiberg-Witten invariants, Kronheimer and Mrówka proved an analog of the adjunction inequality, which states that:

Theorem 1.5 (Kronheimer, Mrówka [9]) *Let X be a smooth, simply connected, compact 4–manifold with $b_+ \geq 2$. If L is a characteristic class with $SW(L) \neq 0$ and S is a compact embedded surface which does not represent a torsion element in homology, and if $S \cdot S \geq 0$, then:*

$$2g - 2 \geq |c_1(L) \cdot S| + S \cdot S$$

In spite of the fact that there are no known examples of manifolds with $b_+ > 1$ that do not satisfy the simple type condition, it is interesting that in the case when the dimension of moduli space $\mathcal{M}_X(L)$ is not zero, one has the following "improvement" of the adjunction inequality:

Theorem 1.6 (Fintushel, Stern [2]) *If L is a characteristic line bundle with nonzero Seiberg–Witten invariant, and X is a smooth manifold with $b_+ \geq 3$, then there exists a constant*

d depending on the $\dim \mathcal{M}_X(L)$ such that if $b_+(X) \geq 2$ and S is represented by an immersed sphere with p positive double points, then either

$$2p - 2 \geq S \cdot S + |S \cdot L| + 2d$$

or:

$$SW_X(L) = \begin{cases} SW_X(L+2S) & \text{if } S \cdot L \geq 0 \\ SW_X(L-2S) & \text{if } S \cdot L \leq 0. \end{cases}$$

In this article we would like to address the following two questions: given a smooth manifold X with a specific configuration of embedded or immersed spheres,

1. Is X of simple type or not?
2. If S is an embedded sphere with negative self-intersection, what are the possible intersection numbers $K \cdot S$ where K is a basic class of X?

Note that Theorem 1.5 does not give an answer to the second question, since it applies only to classes with non-negative self-intersection. After a short review of the blowup formula and its relatives for embedded spheres with negative self-intersection, we restrict our attention to the spheres with self-intersection $p = -2$ or $p = -3$. For a manifold X containing an embedded sphere α with self-intersection $\alpha \cdot \alpha = p$, we define a certain polynomial $C_p(\alpha)$, for which one of the following occurs:

1. If X is of simple type, and $D(C_p(\alpha)) \neq 0$, then there exists a basic class K of X, such that $K \cdot S = p$.
2. If $D(C_p(\alpha)) = 0$, then X is of simple type.

In the last section we give an example of how the existence of embedded 2-spheres affects the Donaldson invariants of X. We prove the following theorem:

Theorem 5.5 *Let the manifold X contain an immersed sphere α with one positive double point and self–intersection 0. Assume also that X contains a cohomology class $f \in \mathrm{H}^2(X)$ such that $f \cdot \alpha = 1$. Then X is of simple type.*

This theorem as such is not new. It follows easily from the result by Kronheimer and Mrówka stating that a manifold X which contains an embedded torus with self-intersection zero (that is essential in cohomology of X) is of simple type. What is new is the fact that in our proof we study only configurations of embedded spheres with negative self-intersection. This approach may lead to removing the assumption of nonnegative self-intersection in the statement of Theorem 1.5.

2 The blowup formula and its extensions

In [4] Fintushel and Stern proved that when e is a class represented by an embedded sphere with self-intersection -1, then

$$D(e^k z) = D(B_k(x) z) \tag{2.1}$$

for every class $z \in \mathrm{H}_2(X)$ such that $z \cdot e = 0$. For brevity from now on we shall omit the class z in the notation, mentioning it explicitly only when necessary. The functions B_k of (2.1), when combined in the power series $B(x,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(x)$ satisfy:

$$B(x,t) = \exp\left(-\frac{t^2 x}{6}\right) \sigma_3(t) \tag{2.2}$$

In the above σ_3 is a particular quasiperiodic Weierstrass sigma-function [1] associated to the $\wp$–function which satisfies the differential equation

$$(y')^2 = 4y^3 - g_2 y - g_3$$

with g_i's given by:

$$g_2 = 4\left(\frac{x^2}{3} - 1\right), \quad g_3 = \frac{8x^3 - 36x}{27}$$

Using the power series notation we can rewrite (2.2) as

$$D(\exp(te)) = D(B(t,x)) \tag{2.3}$$

This formula is known as the blowup formula (since after performing the blowup on the manifold X, the exceptional divisor in $X \# \overline{\mathbb{C}P^2}$ is represented by an embedded sphere with self-intersection -1). There is also a variant of the blowup formula for the twisted Donaldson invariants, that states that:

$$D_e(\exp(te)) = D(S(t,x))$$

where $S(t,x) = e^{-\frac{t^2 x}{6}}\sigma(t)$, and $\sigma(t)$ is the standard Weierstrass sigma–function.

For further computations we shall need explicit formulas for the functions B_k and S_k for some small k. To make the list below one can refer to [4], or one can write a simple *Mathematica* program.

$$\begin{array}{ll} B_2 = 0 & S_1 = 1 \\ B_4 = -2 & S_3 = -x \\ B_6 = 8x & S_5 = 2 + x^2 \end{array} \tag{2.4}$$

As for spheres S of higher negative self-intersection, it turns out that we can not completely eliminate all powers of S from the right hand side of the blowup formula. Instead the following is true (see [3, 10]):

Theorem 2.5 *Let X be an oriented simply connected 4-manifold, which contains an embedded 2-sphere S representing a homology class σ with self-intersection $\sigma^2 \leq -2$. Then there are polynomials $B_{j,k} = B_{j,k}(x)$ depending only on $j = -\sigma^2$, such that for $z \in \mathbf{A}(X) = \mathrm{Sym}_*(\mathrm{H}_0(X) \oplus \mathrm{H}_2(X))$ that is perpendicular to σ*

$$D(\exp(t\sigma)z) = D(\sum_{k=0}^{j} B_{j,k}(x,t)\sigma^k z)$$

if j is even, and:

$$D(\exp(t\sigma)z) = D(\sum_{k=0}^{j-1} B_{j,k}(x,t)\sigma^k z)$$

if j is odd.

Once we know that such universal formulas exist, all we need in order to compute the functions $B_{j,k}$, are the two blowup formulas for -1 curves. We shall give examples of the formulas for $B_{2,k}$ and $B_{3,k}$ in the next two lemmas:

Lemma 2.6 *Let X be an oriented simply connected 4-manifold, which contains an embedded 2-sphere S representing a homology class σ with self-intersection $\sigma^2 = -2$. Then:*

$$D(\exp(t\sigma)) = D(B_{2,0}(t,x) + B_{2,2}(t,x) \cdot \sigma^2)$$

where

$$B_{2,0} = B^2; \text{ and } B_{2,2} = \frac{1}{2}(S)^2.$$

The above result has been claimed by R. Brussee and also has been known to other authors.

Proof: From (2.5) we expect the formula of the form

$$D(\exp(t\sigma)) = D(B_{2,0}(t,x) + B_{2,1}(t,x) \cdot \sigma + B_{2,2}(t,x) \cdot \sigma^2) \tag{2.7}$$

Since every manifold containing a -2–sphere admits a diffeomorphism that on $\mathrm{H}_2(X)$ represents a reflection about σ, then $D(B_{2,1}(t,x) \cdot \sigma) = 0$.

To get the function $B_{2,0}$, consider $\sigma = e_1 + e_2$ contained in $X\#\overline{\mathbb{C}P^2}\#\overline{\mathbb{C}P^2}$, where e_i are the classes represented by exceptional divisors. Then by (2.3) we get:

$$\begin{array}{rcll} D(\exp(t(e_1+e_2)) & = & D(B^2(t,x)) & \text{(by the blowup formula)} \\ & = & D(B_{2,0}(t,x) + B_{2,2}(t,x) \cdot (e_1+e_2)^2) & \text{(by (2.7))} \\ & = & D(B_{2,0}(t,x)) & \text{(by (2.4))} \end{array}$$

Comparing the first and the last equations gives the formula for the function $B_{2,0}$.

In order to get the function $B_{2,2}$, we notice that $(e_1 - e_2) \cdot (e_1 + e_2) = 0$, so we have:

$$\begin{array}{rcl} D\left((e_1-e_2)^2 \cdot \exp(t(e_1+e_2))\right) & = & D((e_1^2 - 2e_1e_2 + e_2^2 \cdot \exp(t(e_1+e_2))) = \\ & = & D(2B'' \cdot B - 2(B')^2) = \\ & = & D\Big((B_{2,0} + B_{2,2} \cdot (e_1+e_2)^2) \cdot (e_1-e_2)^2\Big) = \\ & = & -4D(B_{2,2}) \end{array} \tag{2.8}$$

Since the functions S and B satisfy a relation (see [1]):

$$S^2 = (B')^2 - B''B$$

we get the coefficient $B_{2,2}$ as stated in the theorem. □

In a completely analogous way we can prove the following result for spheres of self-intersection -3:

Lemma 2.9 *Let X be an oriented simply connected 4-manifold, which contains an embedded 2-sphere S representing a homology class σ with self-intersection $\sigma^2 = -3$. Then we have the following formula:*

$$D(\exp(t\sigma)) = D(B_{3,0} + B_{3,1} \cdot \sigma + B_{3,2} \cdot \sigma^2) \tag{2.10}$$

where:

$$\begin{array}{rcl} B_{3,0} & = & B^3 \\ B_{3,1} & = & BSS' - B'S^2 \\ B_{3,2} & = & \frac{1}{2}BS \end{array}$$

Starting from the self-intersection -4 the formulas for $B_{4,i}$'s start to depend on x, and are quite difficult to handle.

3 Basic classes and simple type

The notion of basic classes in Donaldson theory makes sense only for manifolds X that are of simple type. The relation $D(x^2) = 4D$ allows one to simplify the blowup formula, as well as the formulas of Lemmas 2.6 and 2.9. This is due to the fact that the functions $B(x,t)$ and $S(x,t)$ loose their dependence on powers of x that are higher than 2.

We shall derive the information about the basic classes K_i by examining the analytic functions $f(t) = \mathbb{D}_X(t \cdot \alpha)$ for some special classes α. To set the pattern, let us study the case of a manifold X containing an embedded sphere S with self-intersection -1. Then of course X is a blowup of some other manifold, and let e denote the cohomology class represented by the exceptional divisor.

The condition of simple type implies that for any number a

$$D(e^{ax}) = D(\cosh(2a) + \frac{x}{2}\sinh(2a))$$

By applying this to the function $B(x,t)$ we get the following equalities (modulo the kernel of $D(.)$):

$$\begin{aligned} B(x,t) &= e^{-t^2x/6}e^{-t^2/6}\cosh t \\ &= e^{-t^2/6}\cosh t\left(\cosh(\tfrac{-t^2}{3}) + \tfrac{x}{2}\sinh(\tfrac{-t^2}{3})\right) \end{aligned} \tag{3.1}$$

Thus, as a consequence of the blowup formula, we get:

$$\begin{aligned} \mathbb{D}(t\cdot e) &= D\left((1+\tfrac{x}{2})\exp(te)\right) = D\left((1+\tfrac{x}{2})B(x,t)\right) \\ &= D\left((1+\tfrac{x}{2})e^{-t^2/6}\cosh t\left[\cosh(\tfrac{-t^2}{3}) + \tfrac{x}{2}\sinh(\tfrac{-t^2}{3})\right]\right) \\ &= D\left(1+\tfrac{x}{2}\right)e^{-t^2/2}\cosh t \end{aligned} \tag{3.2}$$

On the other hand, from (1.3), we know that:

$$\begin{aligned} \mathbb{D}(t\cdot e) &= \exp(Q_X/2)\textstyle\sum_{i=1}^s a_i\cosh K_i(t\cdot e) \\ &= e^{-t^2/2}\cdot\textstyle\sum_{i=1}^s a_i\cosh\left(t\cdot K_i(e)\right) \end{aligned}$$

Since the functions $\sinh kt$ and $\cosh kt$ for $k \in \mathbb{Z}$ are linearly independent, it follows that if $D(1+\frac{x}{2}) \neq 0$, then $\sum_{j|K_j(e)=1} a_j \neq 0$. In particular it means that there exists a basic class K such that $K \cdot E = 1$. This conclusion as such is also an immediate consequence of the blowup formula, but we shall use the above remarks as a pattern for further considerations.

Now let us assume that X is a manifold containing an embedded sphere S with self-intersection -2. Let α be a cohomology class represented by S. It follows from Lemma 2.6 and formula (3.1) that:

$$\begin{aligned} D\left((1+\tfrac{x}{2})\exp(t\alpha)\right) &= D\left((1+\tfrac{x}{2})e^{-t^2}(\cosh^2 t + \sinh^2 t\cdot\alpha^2)\right) \\ &= e^{-t^2}\cdot\left(D(1+\tfrac{x}{2})\cdot\cosh^2 t + D(\alpha^2+\tfrac{x}{2}\alpha^2)\cdot\sinh^2 t\right) \\ &= e^{-t^2}\cdot\left[D\left(\tfrac{1+\frac{x}{2}}{2}\cdot(1+\alpha^2)\right)\cosh 2t + D\left(\tfrac{1+\frac{x}{2}}{2}\cdot(1-\alpha^2)\right)\right] \end{aligned} \tag{3.3}$$

This last formula allows us to prove the following:

Theorem 3.4 *Let X be a simply connected manifold containing an embedded 2-sphere α with self-intersection -2. Then the following holds:*

1. *If X is of simple type and $D((1+\frac{x}{2})\cdot(1+\alpha^2)) \neq 0$, then there exists at least one basic class K, such that $K\cdot\alpha = 2$.*
2. *If $D((1+\frac{x}{2})\cdot(1+\alpha^2)) = 0$, then X is of simple type.*

Proof: The first part of this theorem follows from (3.3) and linear independence of the functions $e^{-t^2}\cosh nt$. For the second part we notice that from Lemma 2.6 or from [10] we get:

$$D(\alpha^4) = -4D(\alpha^2 x) - 4D \tag{3.5}$$

The assumption in our theorem, when split into degrees of Donaldson invariants modulo 4 reads as the system of conditions:

$$\begin{cases} D(1) = -D(\frac{x}{2}\alpha^2) \\ D(\frac{x}{2}) = -D(\alpha^2) \end{cases}$$

(depending on $b_+(X)$, one of these conditions just states that $0=0$.) Using (3.5) and the first relation, we get $D(\alpha^4) = D(-4\cdot(-2)-4) = D(4)$. On the other hand, when we use the second relation instead, we get $D(\alpha^4) = D(-4\cdot(-\frac{x}{2})\cdot x - 4) = D(2x^2-4)$. Comparing these two equations gives $D(x^2) = 4D(1)$, which is the simple type condition. $\square$

Similar computations can be performed in the situation when X contains an embedded sphere α with self-intersection -3. Then the analog of (3.3) is:

$$\begin{aligned} D\left((1+\tfrac{x}{2})\exp(t\alpha)\right) &= D\left((1+\tfrac{x}{2})e^{-3/2t^2}(\cosh^3 t + \sinh t\cdot\alpha + \tfrac{1}{2}\cosh t\sinh^2 t\cdot\alpha^2)\right) \\ &= e^{-3/2t^2}\left[D\left(\tfrac{1+\frac{x}{2}}{4}\cdot(1+\tfrac{\alpha^2}{2})\right)\cdot\cosh 3t + D\left(\tfrac{1+\frac{x}{2}}{4}\cdot(3-\tfrac{\alpha^2}{2})\right)\cdot\cosh t + \right. \\ &\quad \left. + D\left((1+\tfrac{x}{2})\alpha\right)\cdot\sinh t\right] \end{aligned} \tag{3.6}$$

Thus we have:

Theorem 3.7 *Let X be a 4–manifold containing an embedded sphere S with self-intersection -3. Then:*

1. *If X is a manifold of simple type, and $D((1+\frac{x}{2})\cdot(2+\alpha^2)) \neq 0$, there exists at least one basic class K, such that $K\cdot S = 3$.*
2. *If $D((1+\frac{x}{2})\cdot(2+\alpha^2)) = 0$, then X is of simple type.*

Proof: Again the first part follows from (3.6) and the linear independence of the $\cosh nt$ functions. Similarly to the case of -2-spheres, by using Lemma 2.9 we obtain:

$$D(\alpha^4) = -4D(\alpha^2 x) - 6D$$

The assumption in part 2 of this theorem implies:

$$\begin{cases} D(2) = -D(\frac{x}{2}\alpha^2) \\ D(x) = -D(\alpha^2) \end{cases}$$

giving:

$$\begin{aligned} D(\alpha^4) &= D(-4\cdot(-4)-6) \\ &= D(-4\cdot(-x^2)-6) \end{aligned}$$

which again proves that X is of simple type.

4 Notes on the twisted invariant

Formulas like Lemma 2.6 or Lemma 2.9 are also valid for the twisted Donaldson invariants D_ω for which $\omega \cdot \sigma = 0$. As in the case of embedded spheres with self-intersection -1 the result is different when $\omega \cdot \sigma \neq 0$. In the proof of all reduction formulas one uses the "stretching the neck" argument. Namely, if N is a neighborhood of the embedded sphere with self-intersection p, the boundary of N is the lens space $L = L(p, 1)$. On the collar $L \times (-\epsilon, \epsilon)$ we consider a family of metrics stretching the $(-\epsilon, \epsilon)$ direction to infinity. In the limit we obtain two disjoint manifolds Y and N, each with cylindrical ends. Then every connection A_X on X that contributes to the count of the Donaldson invariant splits as a pair (A_Y, A_N) of connections on Y and N. Both connections A_Y and A_N on the end of the cylindrical tube isometric to $L \times \mathbb{R}$ have a limiting value, which is a flat connection on L. To allow gluing, these limiting values must agree. The gauge equivalence classes of such flat connections form a character variety $\chi(L)$, which in this case is a set of $[\frac{p}{2}] + 1$ points. Given $m \in \chi(L)$ there is a unique $SU(2)$ bundle over N that has the given twisting and chern class m^2/p. When A_Y is an $SO(3)$ connection, in order to extend A_Y over N, we need one extra piece of information. In [7] Kronheimer and Mrówka do this by making a choice of an orientation of the nontrivial factor of $E|L = K \oplus \mathbb{R}$. We shall do it by the choice of the integer lift c of the second Stiefel–Whitney class ω_2. We need to make such a choice anyway in order to orient the moduli space $\mathcal{M}_{k,\omega}(X)$. Since $\mathrm{H}^1(L, \mathbb{Z}) = 0$, it follows from the Mayer–Vietoris sequence:

$$\cdots \to \mathrm{H}^1(L) \to \mathrm{H}^2(X) \to \mathrm{H}^2(Y) \oplus \mathrm{H}^2(N) \to \mathrm{H}^2(L) \to \cdots$$

that every *integer* cohomology class $c \in \mathrm{H}^2(X)$ can be "recovered" from its unique images $c_Y \in \mathrm{H}^2(Y)$ and $c_N \in \mathrm{H}^2(N)$. That is why for two line bundles L_Y over Y and L_N over N (which are given by their Chern classes), we can define a unique line bundle $L = L_Y \# L_N$ over X.

Let us assume that we have given $A_Y \in \mathcal{M}_{k,c_Y}(Y, i)$ where c_Y is the $\mathrm{H}^2(Y)$ component of $c = c_Y + c_N$, and $i \in \chi_{SO(3)}(L) = \{0, 1, \ldots, p\}$. Let L_j denote the complex line bundle over N with $c_1(L_j) = j$. Then the unique reducible $SO(3)$ connection A_j on $L_i \oplus \mathbb{R}$ converges to $\chi(j)$, where the map $\chi : \mathbb{Z} \to \chi(L)$ is a composition of the mod p map from the integers to $\mathbb{Z}_p$, and the projection

$$\pi : \mathbb{Z}_p \to \chi(L)$$

that identifies i and $p - i$. Given the connection A_Y as above, we have two choices of extending A_Y: one by $A_1 = A_Y \# (A_i \oplus \mathbb{R})$ and the other: $A_2 = A_Y \# (A_{p-i} \oplus \mathbb{R})$. If by E_Y we denote the bundle on which A_Y is defined, then the difference between these two extensions, $\delta(c) = c_1(E_Y \# L_j) - c_1(E_Y \# L_{p-j})$ is the integer class that restricts as zero on Y, and evaluates as p on N, thus $\delta(c) = PD(S)$. Thus, as long as $PD(\Sigma) \neq 0 \in \mathrm{H}^2(X, \mathbb{Z}_2)$, the class c forces the choice of the extension of A_Y, which allows us to repeat the proof of the structure theorem for $SO(3)$ connections.

5 Immersed spheres and simple type

In this section we will discuss 4–manifolds containing some simple configurations of embedded or immersed 2–spheres. First we list some formulas that we shall need later.

Lemma 5.1 *Let X contain an embedded 2–sphere σ with self–intersection -4. Let also X contain a class $f \in \mathrm{H}^2(X)$ such that $\sigma \cdot f \equiv 1 \pmod 2$. Then we have the following formulas:*

$$D_f(\sigma^2) = -D_f(x) + 2D_{f+\sigma} \tag{5.2}$$
$$D_f(\sigma^4) = -10D_f(\sigma^2 x) - 9D_f(x^2) + 4D_f \tag{5.3}$$
$$D_f(\sigma^6) = 38D_f(\sigma^4 x) + 471D_f(\sigma^2 x^2) + 432D_f(x^3) + 132D_f(\sigma^2) \tag{5.4}$$

Proof: The coefficients in the above formulas are obtained from the blowup formula applied to particular examples (compare to the proof of Lemma 2.6). □

Next we shall study some specific examples. The first one is described in the following theorem:

Theorem 5.5 *Let the manifold X contain an immersed sphere α with one positive double point and self–intersection 0. Assume also that X contains a cohomology class $f \in \mathrm{H}^2(X)$ such that $f \cdot \alpha = 1$. Then X is of simple type.*

The configuration described in the theorem is shown on the picture below:

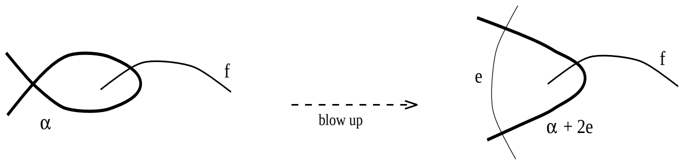

Figure 1. Classes of Theorem 5.5.

Proof: Let us blow up the manifold X at the double point of α. Then the proper transform of α is $\alpha + 2e \in \mathrm{H}^2(X\#\overline{\mathbb{C}P^2})$, where e is an exceptional class in the blowup. This class is represented by an embedded sphere with self-intersection -4. Applying the formulas from Lemma 5.1 to the class $\sigma = \alpha + 2e$, we get:

$$D_f(\alpha^4) = -10D_f(\alpha^2 x) - 9D_f(x^2) + 36D_f \tag{5.6}$$
$$D_f(\alpha^6) = 38D_f(\alpha^4 x) + 471D_f(\alpha^2 x^2) + 432D_f(x^3) + 612D_f(\alpha^2) - 1728D_f(x) \tag{5.7}$$

In the above we have used the fact that $\alpha \cdot \alpha = 0$, and thus also that $\alpha \cdot \sigma = 0$. Since all of these equations are valid for all z perpendicular to σ, by multiplying (5.6) by α^2 we get:

$$D_f(\alpha^6) = -10D_f(\alpha^4 x) - 9D_f(x^2\alpha^2) + 36D_f(\alpha^2) \tag{5.8}$$

Formula (5.8) combined with (5.7) gives:

$$0 = D_f(\alpha^4 x) + 10D_f(\alpha^2 x^2) - 9D_f(x^3) + 36D_f(x) + 12D_f(\alpha^2) \tag{5.9}$$

We can get another expression for $D_f(\alpha^4 x)$ from (5.6) by multiplying it by x. Plugging that into the (5.9) we get:

$$D_f(\alpha^2) = 0 \tag{5.10}$$

Thus (5.8) becomes $D_f(\alpha^6) = -10D_f(\alpha^4 x)$, which applied to (5.7) gives that $D_f(\alpha^4) = 0$. So the equation (5.6) gives the simple type condition for the twisted invariant D_f. It has been shown in [3] that this implies the simple type condition for D. □

References

[1] N. Akheizer, "Elements of the Theory of Elliptic Functions", translated by H. McFaden, A.M.S. Translations of Math. Monographs 79 (1990).

[2] R. Fintushel and R. Stern, *Immersed spheres in 4–manifolds and the immersed Thom Conjecture*, Turkish J. Math. **19** (1995), no. 2, 145–157.

[3] R. Fintushel and R. Stern, *Donaldson invariants of 4- manifolds with simple type*, J. Diff. Geom. **42** (1995), no. 3, 577–633.

[4] R. Fintushel and R. Stern, *The blowup formula for Donaldson invariants* , Annals of Math. (2) **143** (1996), no. 3, 529–546.

[5] P. Griffiths and J. Harris, "Principles of Algebraic Geometry", John Willey and Sons, 1978.

[6] P. Kronheimer and T. Mrówka, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Diff. Geom. **41**, (1995), no. 3, 573–734.

[7] P. Kronheimer and T. Mrówka, *Gauge theory for embedded surfaces*, Topology **34** (1995), no. 1, 37–97.

[8] P. Kronheimer and T. Mrówka, *Recurrence relations and asymptotics for four–manifolds invariants*, Bull. Amer. Math. Soc. (N.S.) **30** (1994), no. 2, 215–221.

[9] P. Kronheimer and T. Mrówka, *The genus of embedded surfaces in the projective plane*, Math. Res. Letters **1** (1994), 797–808.

[10] W. Wieczorek, *Donaldson invariants and embedded 2–spheres*, to appear in Journal für die Reine und Angewendte Mathematik.

[11] E. Witten, *Monopoles and four–manifolds*, Math. Res. Letters **1** (1994), 769–796.