THE DONALDSON INVARIANT AND EMBEDDED 2–SPHERES

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Abstract: In this article we study the Donaldson invariants for smooth 4–manifolds containing an embedded 2–sphere S with an arbitrary negative self–intersection p. We prove that the invariants $D([S]^n)$ for $n > p$, are a combination of invariants $D([S]^i)$ for $0 \leq i \leq p$. We also describe the most efficient way of covering the points that contribute to the Donaldson invariant moduli space by Taubes neighborhoods.

1. INTRODUCTION.

Until the discovery of the Seiberg–Witten invariants the study of smooth 4–dimensional manifolds relied very much on Donaldson invariants. To define these invariants for a simply connected 4–manifold with b_+ odd and bigger or equal to 3 we consider the compactified space $\mathcal{M}_k(X)$ of anti-self–dual (ASD) connections on an $SU(2)$–bundle P over X with $c_2(P) = k$. There is a universal $SO(3)$ fibration $\mathbf{P}$ over $\mathcal{M}_k(X)$ which gives rise to the homomorphism $\mu : \mathrm{H}_i(X) \to \mathrm{H}^{4-i}(\mathcal{M}_k(X))$ given by $\mu(\sigma) = -\frac{1}{4}p_1(\mathbf{P})/\sigma$. This allows one to define Donaldson invariants as linear maps $D_d : \mathbf{A}^d(X) = \mathrm{Sym}^d(\mathrm{H}_0(X) \oplus \mathrm{H}_2(X)) \to \mathbf{R}$ where the elements of $\mathrm{H}_i(X)$ have the degree $\frac{1}{2}(4-i)$ and $\mathbf{A}^d(X)$ is the set of elements of $\mathbf{A}(X) = \mathrm{Sym}_*(\mathrm{H}_0(X) \oplus \mathrm{H}_2(X))$ having degree d. The function D_d assigns to the generator $x \in \mathrm{H}_0(X)$ and the classes $\sigma_1, \dots, \sigma_r \in \mathrm{H}_2(X)$ the number

$$D_d(\sigma_1 \cdots \sigma_r \cdot x^s) = \langle \mu(\sigma_1) \dots \mu(\sigma_r) \cdot \mu(x)^s, [\mathcal{M}_k(X)] \rangle$$

The numbers k, r and s are related through

$$2d = 2r + 4s = 8k - 3(1 + b_+) \tag{1.1}$$

The details of the above constructions can be found in [4] and in [5].

Since the discovery of Donaldson invariants there arose a question about computing them. There has been a large body of work concerning the calculation of these invariants. For example Morgan, Kronheimer and Mrówka [11], and Fintushel and Stern [8] computed the complete Donaldson invariant for simply connected elliptic surfaces. A lot of attention received also the problem of developing some general techniques for computing these invariants, most potent of which is the Mayer-Vietoris type of argument: having the decomposition of the manifold X into $X_1 \bigcup_Y X_2$ and knowing the invariant for the components X_1, X_2 and some properties of Y (which in fact are related to the invariant of the 4–manifold $\mathbf{R} \times Y$), determine the invariant for the manifold X. In a particular case, when X_2 is a neighborhood of an embedded sphere with negative self–intersection, then R. Fintushel and R. Stern provided the following answer in [8]:

Theorem 1.2 *Let X be an oriented simply connected 4-manifold, which contains an embedded 2-sphere S representing a homology class σ with self-intersection $\sigma^2 \leq -2$. Then there are polynomials $B_{j,k} = B_{j,k}(x)$ and $A_{j,k}$ depending only on σ^2, such that for $z \in \mathbf{A}(\sigma^\perp)$*

$$D(\sigma^{2k-1}z) = B_{0,k}D_\sigma(z) + \sum_{j=1}^{k-1} D(\sigma^{2j-1}B_{j,k}(x)z) \qquad \text{if} \qquad \sigma^2 = -(2k+1),$$

and

$$D(\sigma^{2k}z) = A_{0,k}D_\sigma(z) + \sum_{j=1}^{k} D(\sigma^{2j-2}A_{j,k}(x)z) \qquad \text{if} \qquad \sigma^2 = -2k.$$

Here the degrees of the polynomials $B_{j,k}$ are equal to $k-j$, whereas the degrees of $A_{j,k}$ are $k-j+1$. The $A_{0,k}$ and $B_{0,k}$ are constants.

This theorem allowed Fintushel and Stern to give an alternative proof of Kronheimer and Mrówka's theorem stating that the Donaldson invariants on manifolds of simple type depend only on the intersection form Q of X and a finite number of homology classes K_i, called the *basic classes*. In this paper we provide a proof of a slightly more general result than Theorem 1.2:

Theorem 1.3 *Let X be as above and suppose that X contains an embedded S^2 with self-intersection $\sigma^2 = -p \leq -1$. Then for every n, there exist polynomials $B_{p,n,j}(x)$ such that for every $z \in \langle\sigma\rangle^\perp \subset \mathrm{H}_*(X)$ we have:*

$$D(z\sigma^n) = \sum_{j=0}^{p} D(z \cdot B_{p,n,j}(x) \cdot \sigma^j)$$

In the above we assume that $n \geq p+2$ if p is even, and $n \geq p$ if p is odd. Each $B_{p,n,j}$ has the degree d satisfying the inequality $2d + j \leq n$.

Some comments are necessary about the above theorem. Although Theorem 1.2 is not within the range of the Theorem 1.3, one can easily adjust the proof of (1.3) to get (1.2). Two examples at the end of this article show how to do it. Next, our proof is independent of the compactifications of certain open sets in the moduli spaces provided by C. Taubes [20] that are used in [8]. Moreover, our technique allows us to compute the polynomials $B_{p,n,j}$ in certain cases. In fact the formula

$$D(z\sigma^4) = -4D(zx\sigma^2) - 4D(z)$$

for $\sigma \cdot \sigma = -2$ was a crucial component of the Fintushel–Stern proof of the blow–up formula [9], that gives the complete description of the functions $B_{1,n,j}$. Using the observation that when e is the class of an embedded sphere with $e \cdot e = -1$ then $2e$ is also represented by an embedded sphere, one can derive the formula for the functions $B_{2,n,j}$ as well. For a given $z \in \mathrm{H}_2(X)$ only half of the terms in (1.3) are nontrivial. Due to the restriction (1.1), either $D(\sigma^{even} \cdot z)$ or $D(\sigma^{odd} \cdot z)$ is always zero. Also, when $p = 1$ or 2, X admits an orientation–preserving diffeomorphism that induces a map on $\mathrm{H}_2(X)$ sending σ onto $-\sigma$ and which is identity on $\sigma^\perp$. This implies that $D(\sigma^{odd}z) = 0$ for all z.

In order to prove Theorem 1.3, we split the manifold X into N, the neighborhood of the embedded sphere with negative self–intersection, and the remaining part Y. These spaces are connected along the lens space $L = L(p,1)$. Section 2 provides a review of Morgan, Mrówka, and Ruberman's results on gluing ASD connections together. Roughly speaking, by stretching the cylinder $L \times (-\epsilon, \epsilon)$ we obtain open sets U_i in the moduli space for the manifold X that can be described in terms of moduli spaces of Y and N. In order to combine these open sets back into a global one, we need to understand how these sets overlap. Section 3 describes the moduli spaces of connections on the cylinders $\mathbf{R} \times L(p,1)$, which are essential components of the sets $U_i \cap U_j$. In section 4 we describe the most efficient way of chosing the sets U_i, which cover those points of the moduli space over X which contribute nontrivially to the Donaldson invariant. We introduce a partial order in the set $\{U_i\}$ that will enable us to make an inductive argument in the proof of the main theorem. The next section is devoted to computations inside the sets U_i. In the same section, by using a relative Mayer–Vietoris sequence, we also show that the computations in U_i's done separately suffice for proving a global result. Finally, in section 6 we prove the main result. The techniques developed in this paper have wider application than just proving the existence of a general formula. In the final section we compute three examples of particular formulas. One of such formulas has been used by [9] as a necessary initial condition for their differential equation defining the function $B_{1,j,k}(X)$ in the blowup formula. All of the examples computed here agree with the known invariants for elliptic surfaces.

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2. REVIEW OF THE GLUING THEOREM.

The main technique used in the proof of Theorem 1.3 is stretching the neck. A smooth manifold containing a sphere σ with self–intersection $-p$ we can written as $Y \underset{L}{\cup} N$, where N is the tubular neighborhood of σ, and $L = L(p,1)$. In X we can identify a set isometric to $L \times (-\epsilon, \epsilon)$ and take a sequence of metrics on X that would stretch the length of $(-\epsilon, \epsilon)$ to infinity. In the limit we get two manifolds with cylindrical end, which we shall denote by the same letters Y and N. Each of these spaces contains a cylinder isometric to $\mathbf{R} \times L$. There are corresponding moduli spaces $\mathcal{M}_k(Y)$ of ASD connections on such manifolds with similar properties to the ones on the closed manifolds. (For details see for example, [15].) The novum is that the energy (the second Chern class of the bundle on which the connections are defined) does not have to be an integer. By $\mathcal{M}_k^x(Y)$ we shall denote the space of the *based* ASD connections. The other feature of these moduli spaces is that there exists a smooth map $\partial^o : \mathcal{M}_k^x(Y) \to \mathcal{R}(L)$ from the based moduli space to the representation variety $\mathcal{R}(L) = \mathrm{Hom}(\pi_1(L), SU(2))$. This map descends to $\partial : \mathcal{M}_k(Y) \to \chi(L)$, where $\chi(L)$ is *the character variety* of L. These maps associate to every gauge equivalence class of ASD connection ω (or the class of $[\omega, g]$, where $g \in SU(2)$ is the basing) the limit $\lim_{t\to\infty}[\omega|_{t\times L}]$. This limit is the gauge equivalence class of a flat connection on L. In the particular situation when $L = L(p,1)$ the character variety is just the set of $[\frac{p}{2}] + 1$ isolated points. For the future reference we give the following definition:

Definition 2.1 We will refer to the elements 0 and $p/2 \in \chi(L)$ as *trivial* elements of the character variety. Thus for p odd there is only one trivial element. The remaining elements of $\chi(L)$ we will call *nontrivial.*

The representation variety consists of an isolated point or two points corresponding to trivial elements in $\chi(L)$, and a copy of S^2 corresponding to every nontrivial element of $\chi(L)$. For given $m \in \chi(L)$ define $\mathcal{M}_k(Y,m)$ as $\partial^{-1}(m)$. The corresponding space of based connections is denoted by $\mathcal{M}_k^x(Y,m)$.

Our goal is to describe some open sets in $\mathcal{M}_k(X)$ in the terms of moduli spaces on Y and N. For that purpose define the set $\mathcal{M}^x_{k_Y}(Y,T,m)$ as containing those ASD connections on Y, whose energy on the cylinder $[T,\infty)\times L$ is small. (The term "small" can be made precise by using the ε_0 defined in Corollary 4.3.3 of [15].) Same definition holds for N instead of Y. Then consider the fibered product

$$U^{m,x}_{k_Y,k_N} = \mathcal{M}^x_{k_Y}(Y,T_Y,m) \underset{\partial^o}{\times} \mathcal{M}^x_{k_N}(N,T_N,m) \tag{2.2}$$

If we cut the spaces with infinite ends Y and N at the length l past the defined limits T_Y and T_N, then we can glue these manifolds along a set isometric to $[T,T+l]\times L$ obtaining the closed manifold X_l. Given a pair of connections (ω^Y,ω^N) with the help of cut–off functions one can define the (usually not ASD) connection $\gamma_l(\omega_Y,\omega_N)$ on X_l. (For details see [14, 15].) The following theorem relates the connection $\gamma_l(\omega_Y,\omega_N)$ to a unique ASD connection on X:

Theorem 2.3 (Propositions 4.1.1 and 4.2.1 in [14]) *There is a constant l_0 depending only on the sets $\tilde U^m_{k_Y,k_N}$ such that for every $([\omega_Y,\omega_N])\in U^{m,x}_{k_Y,k_N}$ and $l\geq l_0$ there is a unique solution $u=u(\omega_Y,\omega_N)$ to the equation*

$$F^+(\gamma_l(\omega_Y,\omega_N)+u)=0$$

Moreover, the assignment $([\omega_Y,\omega_N])\to\gamma_l(\omega_Y,\omega_N)+u$ defines a unique smooth map γ^o_l : $U^{m,x}_{k_Y,k_N}\to\mathcal{M}^x_{k_Y+k_N}(X_l)$ that is a diffeomorphism onto an open subset of $\mathcal{M}^n_{k_Y+k_N}(X_l)$. This diffeomorphism factors through to $\gamma_l : U^m_{k_Y,k_N} = U^{m,x}_{k_Y,k_N}/SO(3)\to\mathcal{M}_{k_Y+k_N}(X_l)$.

The above theorem can be used also for defining the reverse process: we can define open sets in $\mathcal{M}_k(X)$ that are homeomorphic to the sets $U^m_{k_Y,k_N}$. To define these open sets let us fix a cylinder $C_d=(-d,d)\times L$ for some $d\geq l_0$. This cylinder splits X_l into two connected components, to which we shall refer as the Y–side and N–side. Then define $\mathcal{M}_{k_Y,k_N}(X_l,C_d)$ to be the set of those $\omega\in\mathcal{M}_{k_Y+k_N}(X_l)$ that have small energy on the cylinder C_d, the energy on the on the Y–side is close to k_Y, and is approximately k_N on N–side. (t The terms "small" and "approximately" again refer to the number ε_0 used in the domain of gluing theorem (2.3).) We have the following lemma:

Lemma 2.4 *Assume that l satisfies the assumption of the gluing theorem. Then if the set $\mathcal{M}^x_{k_Y,k_N}(X_l,C_d)$ is nonempty, then there exists a set $\mathcal{X}\subset\chi(L)$ and a smooth map*

$$\mathcal{M}^x_{k_Y,k_N}(X_l,C_d)\overset{\rho}{\to}\coprod_{m\in\mathcal{X}}U^{m,x}_{k_Y,k_N}$$

Proof: Since the character variety $\chi(L)$ is a discrete set of points, for every connection ω from $\mathcal{M}_{k_Y,k_N}(X_l, C_d)$ we can find a unique gauge equivalence class of flat connection $\Gamma_m \in \chi(L)$ such that Γ_m is the closest to $\omega|_{\{0\}\times L}$ where $\{0\} \times L \subset C_d$. (The distribution of energy (k_Y, k_N) puts restrictions on possible $m \in \chi(L)$. In fact $CS(m) \equiv k_Y \equiv -k_N \mod \mathbf{Z}$, where CS is Chern–Simon invariant, but this is not essential in this proof.) Let ω_m be a corresponding flat connection on the infinite cylinder $\mathbf{R} \times L$. Then we can define two approximate ASD connections $\gamma[\omega|_Y, \omega_m]$ on Y and $\gamma[\omega|_N, \omega_m]$ on N, that according to Theorem 2.3 can be perturbed to a pair of ASD connections $[\omega_Y, \omega_N] \in U^m_{k_Y,k_N}$. □

The following theorem describes the image of "ungluing" map ρ:

Theorem 2.5 (see Theorem 4.3.1 of [14]) *For l long enough, the set $U^m_{k_Y,k_N}$ is in the image of ungluing.*

3. QUANTIZATION OF THE MODULI SPACE.

The goal of this section is to cover the intersection

$$\mathcal{M}_{k_0}(X) \cap V_z \cap \bigcap_{i=1}^{n} V_{\sigma_i} \tag{3.1}$$

with the minimal number of open sets of the form $U^m_{k_Y,k_N}$. As we indicated before, we shall stretch the metric on the cylinder $(-\delta, \delta) \times L \subset X$ so as to identify it with $(-n, n) \times L$ for some big n. Next we would like to fix disjoint cyliders $R_i \cong [0, l] \times L$ in $(-n, n) \times L$ so that every connection in (3.1) allows an ungluing along one of R_i in the sense of Lemma 2.4. The question is: how many sets R_i do we need? It turns out that the most fruitful approach is by studying *dimensions* of the moduli spaces on the cylinders $\mathbf{R} \times L$ that result from splitting X at more than one region R_i. That is why we shall start from the description of these moduli spaces.

3.1. THE DESCRIPTION OF INSTANTONS ON $\mathbf{R} \times L$.

Certain $SU(2)$ bundles over $\mathbf{R} \times L$ can be obtained from $SU(2)$ bundles E over a sphere S^4 by dividing by a $\mathbf{Z}_p$ action. It turns out that the condition for bundles to arise in this way is closely related to the existence of ASD connections on these bundles. Following [6], we can describe the $\mathbf{Z}_p$-invariant bundles over S^4 as

$$(D^4_- \times SU(2), T_2) \bigcup_F (D^4_+ \times SU(2), T_1) \tag{3.2}$$

where $\xi = \mathrm{e}^{\frac{2\pi i}{p}}$ acts on $D^4_\pm \subset \mathbf{C}^2$ via $\xi(z_1, z_2) = (z_1\xi, z_2\xi)$, and for $g \in SU(2)$,

$$T_1(g) = g \cdot \begin{pmatrix} \exp \xi m & 0 \\ 0 & \exp \bar{\xi} m \end{pmatrix}, \text{ and } T_2(g) = g \cdot \begin{pmatrix} \exp \xi m' & 0 \\ 0 & \exp \bar{\xi} m' \end{pmatrix}$$

We will refer to the numbers m and m' as the *weights* of the $\mathbf{Z}_p$ action. The transition function $F : S^3 \times SU(2) \to S^3 \times SU(2)$ of (3.2) can be written as $F(x, g) = (x, f(x)g)$ for $f : S^3 \to SU(2)$. The $\mathbf{Z}_p$ equivariance of F means that $f(x\xi) = T_2^{-1} f(x) T_1$. Notice that the degree k of the map f is equal to the second Chern class of the bundle E. We denote the bundle obtained from the above construction divided by a Z_p action by $E(k, m, m')$. [6] adapted to the case of $SU(2)$ bundles over $\mathbf{R} \times L(p, 1)$ gives a construction of $\mathbf{Z}_p$ equivariant bundles with weights (m, m') and degree $k = m'^2 - m^2$. Let us now denote by $\mathcal{M}_{k/p}(\mathbf{R} \times L, [m, m'])$ the moduli space of ASD connections on the bundle $E(k, m, m')$. Before quoting the results of [1] describing moduli spaces of ASD connections on $E(k, m, m')$, we need to comment on orientations. The construction of ASD connections is based on the identification of S^4 with quaternions $\mathbf{H}$; thus when $\mathbf{R} \times L$ is the boundary of a complex manifold M, the orientations in [1] agree with the convention that the weight m is from "the manifold side" and m' is from the other side. In our particular application we are studying the spaces $\mathbf{R} \times L$ as the ends of the space $N = \overline{\mathbf{C}P^2}$, thus we have to reverse orientations coming from the complex structures. We can do it in two ways: either by changing the convention about the order of weights or by changing $L(p, q)$ to $L(p, -q)$. In this paper we have chosen the second way, thus even though the boundary of N is $L(p, -1)$, while quoting Austin's paper we fix $q = +1$. The main construction in [1] (Lemma 5.1) gives a description of ASD connections on the bundles $E(k, m, m')$ for which there exists a solution (a, b) to a system:

$$\begin{aligned} a &\equiv m + m' \pmod{p} \\ b &\equiv m' - m \pmod{p} \\ a \cdot b &= k \end{aligned} \tag{3.3}$$

These bundles are called S^1 *equivariant.* (The reason for this name is that the $\mathbf{Z}_p$ action on these bundles extends to an S^1 action). To state the general result we need one more definition:

Definition 3.4 A $\mathbf{Z}_p$ equivariant bundle $E(k, m, m')$ is a composite of $\mathbf{Z}_p$ equivariant bundles $\{E(k_i, m_i, m'_i)\}_{i=1}^n$ if and only if for all $i = 1, 2, \ldots, n$ we have:
$k_i > 0;\quad k = \sum_i k_i;\quad m \equiv m_1 \pmod{p};\quad m'_i \equiv m_{i+1} \pmod{p};\quad m'_n \equiv m' \pmod{p}$

When two bundles $E_1 = E(k_1, m_1, m'_1)$ and $E_2 = E(k_2, m_2, m'_2)$ have $m'_1 \equiv m_2 \pmod{p}$, then there is a $\mathbf{Z}_p$ equivariant isomorphism of $(E_1|_{D^4_+}, T_1) \equiv (E_2|_{D^4_-}, T_2)$. Using this isomorphism we can construct a bundle with weights m_1 and m'_2, having $k_1 + k_2$ as its Chern class. Thus when the bundles $\{E(k_i, m_i, m'_i)\}_{i=1}^n$ satisfy the formal requirements of Definition 3.4, then we can construct a topological bundle $E(k, m, m')$.

Lemma 3.5 (Compare Lemma 5.2 in [1].) *A $\mathbf{Z}_p$ equivariant bundle supports a $\mathbf{Z}_p$ equivariant ASD connection if and only if it is a composite of S^1 equivariant bundles.*

The minimal amount of energy on a nontrivial moduli space on $\mathbf{R} \times L$ is $\frac{1}{p}$. The next lemma provides the first limit on possible distributions of energies:

Lemma 3.6 *For fixed m and m' let k be the minimal amount of energy for which the bundle $E(k, m, m')$ supports an ASD instanton. Then*

$$k = \begin{cases} m'^2 - m^2 & \text{if } m' > m \\ m'^2 - m^2 + p(m - m') & \text{if } m' < m \end{cases}$$

Proof: As $m, m' \leq p/2$, then $(a, b) = (m + m', m' - m)$ is the smallest nonegative solution to (3.3) in the case $m' > m$, and $(a, b) = (m - m', p - m - m')$ is the smallest solution in the case $m' < m$. The product of these numbers gives the required result for S^1 equivariant bundles. Now assume that $E(k, m, m')$ is a composite of $\{E(k_i, m_i, m'_i)\}_{i=1}^n$. The in the case of $m' > m$ we have:

$$k = \sum k_i \geq m'^2_n - m_1^2 + \sum_{i \in I} p(m_i - m'_i) \geq m'^2_n - m_1^2$$

Here I is the set of those indexes i for which $m_i > m'_i$. In the case $m' < m$ we have: $k \geq m'^2_n - m_1^2 + \sum_{i \in I} p(m_i - m'_i) \geq m'^2_n - m_1^2 + \sum_{i=1}^n p(m_i - m'_i) = m'^2_n - m_1^2 + p(m_1 - m'_n)$ □

The classical (by now) results of [15] and [1] give us the following

Theorem 3.7 *The dimension of $\mathcal{M}_\kappa(Y, m)$ is equal to*

$$8\kappa - \frac{3}{2} \cdot (\sigma(Y) + \chi(Y)) - \frac{1}{2} + \frac{8m^2}{p} + 1 - 4m$$

when m is nontrivial, and

$$8\kappa - \frac{3}{2} \cdot (\sigma(Y) + \chi(Y)) - \frac{3}{2}$$

if m is trivial. Similarly

$$\dim \mathcal{M}_\kappa(N, m) = \begin{cases} 8\kappa - 3 + 4m - \frac{8m^2}{p} & \textit{if } m \textit{ is nontrivial} \\ 8\kappa - 3 & \textit{if } m \textit{ is trivial.} \end{cases} \tag{3.8}$$

One can notice that the first formula in (3.8) gives $8\kappa - 3$ for $m = 0$ or $\frac{p}{2}$, so it is valid in both cases. It is interesting that, according to the dimension formula given by Austin in [1], we have:

$$\dim \mathcal{M}_\kappa(N, m) = \dim \mathcal{M}_\kappa(\mathbf{R} \times L, [0, m])$$

Thus, in particular, we have:

$$\dim \mathcal{M}_{m^2/p}(\mathbf{R} \times L(p, 1), [0, m]) = 4m - 3 \quad \text{for } m \leq p/2 \tag{3.9}$$

In order to formulate the result which allows us to quantize moduli spaces on the cylinders, we need to notice a simple fact connected with gluing. This fact has also been mentioned in [7]:

Lemma 3.10 *Let $\mathcal{M}^o = \mathcal{M}^o_\kappa(X,m)$ for a nontrivial element m of the character variety. Then the base point fibration $\mathcal{M}^o \to \mathcal{M}$ reduces, i.e. there exists an S^1-bundle $\mathcal{Q}$ such that $\mathcal{M}^o = \mathcal{Q} \underset{S^1}{\times} SO(3)$.*

Proof: Consider the map $\partial^o : \mathcal{M}^o \to S^2 \subset \mathcal{R}(L)$. This map restricts over each fiber as $\partial^o|_{SO(3)} : SO(3) \to S^2$ that sends $g \to g\xi g^{-1}$. Thus for fixed point $pt \in S^2$, $(\partial^o)^{-1}(pt)$ is an S^1 bundle $\mathcal{Q}$ over S^2 for which $\mathcal{M}^o = \mathcal{Q} \underset{S^1}{\times} SO(3)$. □

The moduli spaces on the cylinders $\mathbf{R} \times L$ have two boundary value maps

$$\partial^o_\pm : \mathcal{M}^o_\kappa(\mathbf{R} \times L(p,1), [m,m']) \to \mathcal{R}(L)$$

with $\kappa \in \mathbf{Z}[\frac{1}{p}]$. Let us recall that according to our convention on the cylinders we consider as positive the direction from N to Y. Thus ∂^o_- points toward N, and ∂^o_+ points toward Y. With this understood, define $\mathcal{M}^G_\kappa(\mathbf{R} \times L(p,1), [m,m'])$ to be $(\partial^o_-)^{-1}(pt)$ for any $pt \in$ Im (∂^0_-). Thus when m is a trivial element of $\chi(L)$, then $\mathcal{M}^G_\kappa(\mathbf{R} \times L(p,1), [m,m'])$ is the based moduli space $\mathcal{M}^o_\kappa(\mathbf{R} \times L(p,1), [m,m'])$. When m is nontrivial, then according to Lemma 3.10 the space $\mathcal{M}^G_\kappa(\mathbf{R} \times L(p,1), [m,m'])$ is an S^1 reduction of the base–point fibration. This space should be thought of as the gluing parameter bundle over an unbased moduli space. The next lemma shows that these moduli spaces provide good quantization of the whole moduli space:

Lemma 3.11 $\dim \mathcal{M}^G_\kappa(\mathbf{R} \times L(p,1), [m,m']) \geq 4 \cdot |m - m'|$.

Proof: This is a direct consequence of Lemma 3.6 describing the lowest energy on the moduli space $\mathcal{M}^G_\kappa(\mathbf{R} \times L(p,1), [m,m'])$ and the dimension formula (3.9).

Case 1: $m' > m$. The moduli space $\mathcal{M}_{m'^2/p}(\mathbf{R} \times L(p,1), [0,m'])$ has a boundary corresponding to a splitting

$$\mathcal{M}^o_{m^2/p}(\mathbf{R} \times L(p,1), [0,m]) \underset{\partial^o}{\times} \mathcal{M}^o_{(m'^2-m^2)/p}(\mathbf{R} \times L(p,1), [m,m'])$$

where all of the moduli spaces involved have the minimal energy required for each tunneling. From (3.9) we have:

$$\dim \mathcal{M}_{m'^2/p}(\mathbf{R} \times L(p,1), [0,m']) = 4 \cdot m' - 3 = 4 \cdot m - 3 + \dim \mathcal{M}^G_\kappa(\mathbf{R} \times L(p,1), [m,m'])$$

from which the lemma follows.

Case 2: $m' < m$. From the index formula we get that $\dim \mathcal{M}_m(\mathbf{R}\times L(p,1), [0,0]) = 8m - 3$. This space allows a splitting $\mathcal{M}^o_{m^2/p}(\mathbf{R}\times L(p,1), [0,m]) \underset{\partial^o}{\times} \mathcal{M}^o_{m(p-m)/p}(\mathbf{R}\times L(p,1), [m,0])$. Analyzing the dimensions of both spaces we get:

$$8m - 3 = 4m - 3 + \dim \mathcal{M}^G_{m(p-m)/p}(\mathbf{R} \times L(p,1), [m,0])$$

Thus $\dim \mathcal{M}^G_{m(p-m)/p}(\mathbf{R} \times L(p,1), [m,0]) = 4m$.

Now consider the space $\mathcal{M}_{(pm-m^2)/p}(\mathbf{R}\times L(p,1),[m,0])$ and its splitting

$$\mathcal{M}^o_{\frac{m'^2-m^2}{p}+m-m'}(\mathbf{R}\times L(p,1),[m,m']) \underset{\partial^o}{\times} \mathcal{M}^o_{\frac{pm'-m'^2}{p}}(\mathbf{R}\times L(p,1),[m',0])$$

Again the energies have been chosen according to Lemma 3.6 so as to be the smallest ones. Analyzing the dimensions of the involved spaces we get the equation:

$$4m = \dim \mathcal{M}^G_\kappa(\mathbf{R}\times L(p,1),[m,m']) + 4m',$$

which concludes the proof of the lemma. □

3.2. COVERING MODULI SPACE WITH OPEN SETS.

Now we are ready to list all open sets that cover the intersection (3.1). To fix notation, let us assume that the class $z\in \mathbf{A}(Y)$ has degree d, and

$$\dim \mathcal{M}_{k_0}(X) = 2\,(d+n)$$

Since we are using a generic metric on X, the intersection of divisors on Y–side is transverse, so the intersection $U^m_{k_Y,k_N}\cap V_z\cap \bigcap_{i=1}^n V_{\sigma_i}$ (see (2.2) for the definition of $U^m_{k_Y,k_N}$) is empty if $\dim\,\mathcal{M}_{k_Y}(Y,m)\le 2d$. This implies that $\dim\,\mathcal{M}_{k_N}(N,m)$ must not exceed $2n-1$ if m is nontrivial nor $2n-3$ if m is trivial. In order to choose the sets $U^m_{k_Y,k_N}$ in the most efficient way, let us define the set

$$\begin{aligned}\mathcal{J} \;=\; &\{(m,k)\mid m\in\chi(L),\ k\in \mathbf{Z}[\tfrac{1}{p}], \mathcal{M}_k(N,m)\neq\emptyset,\\ &\text{and } 0<\ \dim\mathcal{M}_k(N,m)+h_m\le 2n\}\end{aligned}$$

Here h_m is the sum of h^i_m, the dimensions of cohomology groups $\mathrm{H}^i(L,\mathrm{ad}\,m)$ for $i=0,1$. Note that the bound on the dimension of the spaces $\mathcal{M}_k(N,m)$ implies that the set $\mathcal{J}$ is finite. In $\mathcal{J}$ we define a partial order by saying that $(m_1,k_1)\le(m_2,k_2)$ if there is a nonempty moduli space $\mathcal{M}_{k_2-k_1}(\mathbf{R}\times L,[m_1,m_2])$. We set $\bar{\mathcal{J}}=\mathcal{J}\cup\{(0,0)\}$.

Definition 3.12 To every element $(m,k)\in\bar{\mathcal{J}}$ assign its *degree* $\deg\,(m,k)$ to be the length of the longest chain in $\bar{\mathcal{J}}$ from $(0,0)$ to (m,k) linearly ordered by $\le$. Define $\deg(\mathcal{J}) = \max_{x\in\mathcal{J}}\,\deg(x)$.

The next lemma shows how to compute the above degrees.

Lemma 3.13 *For every $(m,k)\in\mathcal{J}$ we have the following relation:*

$$4\cdot(\deg\,(m,k)-1)+1 = \dim\mathcal{M}_k(N,m)$$

Proof: Set $\alpha = \deg\,(m,k)$. The definition of α implies that the moduli space $\mathcal{M}_k(N,m)$ has a boundary component corresponding to a fibered product of $(\alpha-1)$ moduli spaces $\mathcal{M}_{k_j}(\mathbf{R}\times L,[m_j,m_j'])$ and the last α^{th} space $\mathcal{M}_{k_N}(N,m_N)$. Thus from (3.11) we have: dim $\mathcal{M}_k(N,m) \geq 4(\alpha-1)+1$.

We prove the inequality in the opposite direction by induction on the degree of elements of $\mathcal{J}$. Lemma 3.6 gives us the following inequalities in $\mathcal{J}$:

$$(m,k) \leq (m+1, k+\frac{2m+1}{p}) \leq (m,k+1)$$

Thus deg $(m,k+1) \geq$ deg $(m,k)+2$. Using the dimension formula (3.8) we get:

$$\begin{array}{rcl} \dim\mathcal{M}_{k+1}(N,k+1) & = & \dim\mathcal{M}_k(N,k)+8 = 4(\deg\,(m,k)-1)+1+8 \\ & \leq & 4(\deg\,(m,k+1)-3)+1+8 = 4(\deg\,(m,k+1)-1)+1 \end{array}$$

□

Corollary 3.14 *Let Γ be a graph representing the set $\mathcal{J}$ with the relation $\leq$. Then $(m,k) \leq (m',k')$ are immediate neighbors in the graph Γ if and only if $m' = m \pm 1$ and*

$$k'-k = \begin{cases} \frac{1}{p}\cdot[2m+1] & \text{if } m' = m+1 \\ \frac{1}{p}\cdot[-2m+1]+1 & \text{if } m' = m-1 \end{cases}$$

Proof: The fact that $(m,k) \leq (m',k')$ for m' and k' defined above follows from Lemma 3.6. The Lemma 3.13 gives us:

$$\begin{array}{rcl} \deg(m',k')-\deg(m,k) & = & \frac{1}{4}\left[\dim\mathcal{M}_{k'}(N,m')-\dim\mathcal{M}_k(N,m)\right] \\ & = & \frac{1}{4}\dim\mathcal{M}^G_{k'-k}(\mathbf{R}\times L,[m,m']) = 1 \end{array} \tag{3.15}$$

thus there are no other points of $\mathcal{J}$ between (m,k) and (m',k'). □

Now for each element of $\mathcal{J}$ we wish to assign an open set in the moduli space $\mathcal{M}_k(X)$. Let us fix $s = \deg(\mathcal{J})$. Notice that by changing the metric on X, we can identify in X a subset isometric to a cylinder $[-n,n]\times L$ for an arbitrary large n. We can adjust its length n so as to have a collection of disjoint subcylinders $C_1^0,\ldots,C_t^0,R_1,C_1^1,\ldots,C_t^1$, $R_2,C_1^2,\ldots C_t^2,\ldots R_s,C_1^s,\ldots C_t^s$ satisfying the following:

1. For each C_i and R_i there is some $r\in[-n,n]$ such that these cylinders are equal to $[r,r+l]\times L$, where l is the length satisfying the assumptions of Theorem 2.4;
2. For every connection ω with energy k on X there are $s+1$ numbers $i_0,i_1,i_2,\ldots,i_s$ $(1\leq i_r\leq t)$ such that the energy of the connection ω on the cylinder $C_{i_r}^r$ is less than the number ε_0, that was used in defining the set $U^{m,x}_{k_Y,k_N}$ in (2.2).

Now we are ready to state the following definition:

Definition 3.16 For every element $(m,k)\in\mathcal{J}$ we define an open set $U_k^m \subset \mathcal{M}_{k_0}(X)$ as the set of all ω that satisfy:

1. The energy of ω on R_i is less than ε_0, where $i = deg(m,k)$.

2. The energy of ω on the N–side of R_i is close to k (see the discussion proceeding the formula (2.2).

3. If (r,n) is a point in R_i then the restriction $\omega|_{\{r\}\times L}$ is closest to the character m.

Having this assignment in mind we shall understand that the collection of open sets $\{U_k^m\}$ has also defined a partial order "$\leq$" and $\deg(U_k^m)$ makes sense as well. Each set U_k^m has uniquely assigned region R_i with $i = \deg(m,k)$ and let us choose one point $x_i \in R_i$. According to Lemma 2.5 each U_k^m can be described as the fibered product:

$$\mathcal{M}^{x_i}_{k_X-k}(Y,m) \underset{\partial}{\times} \mathcal{M}^{x_i}_k(N,m)/SO(3) \tag{3.17}$$

Thus every connection in U_k^m can be written as $\omega = [\omega_Y, \omega_N]$. In general, if a given connection ω allows the representation (3.17) for some i, then we say that ω allows splitting at the region R_i. In the next lemma, we shall prove that the sets U_k^m suffice for covering the intersection of divisors $V_z \cap \bigcap_{i=1}^{n} V_\sigma$. As the proof of that lemma requires splittings of moduli space at multiple regions $R_1, R_2, \ldots, R_r$ on the cylinder $[-n,n]\times L$, we need to be more precise about the notation of cylindrical–end manifolds Y and N. Thus by Y and N we shall mean finite cylinder manifolds ending at the first splitting regions (including these regions). For a given region R we shall mean by the terms Y–side or N–side the corresponding connected components of the manifold X cut by R. (See the picture below):

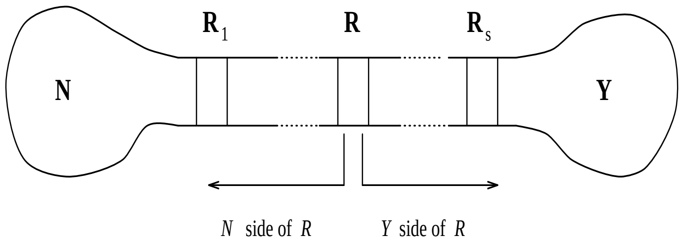

Figure 1. The splitting of the manifold X.

Lemma 3.18 *Every connection that belongs to the intersection $\mathcal{M}_{k_0}(X) \cap V_z \cap \bigcap_{i=1}^{n} V_\sigma$ is contained in one of the sets U_i^m, $(m,i) \in \mathcal{J}$. Moreover deg $(\mathcal{J}) \leq \frac{n+1}{2}$.*

Proof: First we shall prove that $s = \deg \mathcal{J}$ is a sufficient number of the splitting regions. Assume that ω is a connection that does not allow splitting at any of the regions $R_1, \ldots, R_s$. According to the definition of the length l, there are cylinders $C^r_{i_r}$, $r = 0,1,\ldots,s$ such that ω has energy less than ε_0 in these regions, thus it allows the splitting at $C^r_{i_r}$. Let m_r denote the gauge equivalence class of the limiting flat connection on $C^r_{i_r}$, and k_r the energy on the N–side of $C^r_{i_r}$. As $\omega \in \mathcal{M}_{k_0}(X) \cap V_z \bigcap_{i=1}^{n} V_\sigma$, then for every r, the dimension bound $\dim \mathcal{M}_k(N, m_r) \leq 2n$, defining the set $\mathcal{J}$, is satisfied, thus the pair $(m_r, k_r) \in \mathcal{J}$ for every r. By Lemma 2.5, ω can be represented as a fibered product of connections $[\omega_N, \omega_1, \ldots, \omega_s, \omega_Y]$.

The first connection ω_Y lies in a moduli space of nonnegative dimension. If one of connections ω_i defined on $[l_i, l_{i+1}] \times L \supset R_i$ was flat, then it would allow the obvious splitting along R_i as $\omega_i = [\omega_{i,-}, \omega_{i,+}]$. In here $\omega_{i,\pm}$ are the restrictions of ω_i onto different connected components of $[l_i, l_{i+1}] \times L \setminus R_i$. As each ω_j is nonflat, then we have a sequence of strict inequalities in $\mathcal{J}$

$$(m_N, k_N) < (m_1, k_1) < \cdots < (m_r, k_r)$$

from which we get that the last pair (m_r, k_r) has the degree greater than or equal to $s+1$, thus contradicting the definition of s.

Next assume that ω allows splitting at R_i with k_i being its energy on the $N-$ side and m_i being the nearest flat connection on R_i in the sense of Definition 3.16. We still need to prove that for some i there is the equality $\deg(m_i, k_i) = i$. So assume there is an ω which satisfies: for every $i = 1, 2, \ldots, s$ either ω does not split at R_i or $\deg(m_i, k_i) \neq i$ for (m_i, k_i) defined by a splitting at R_i. Consider first the case when ω splits at R_i with $\deg(m_i, k_i) < i$. We claim that there exists another region R_j for which the equality $\deg(m_j, k_j) = j$ holds. This is proved by induction with respect to i. For $i = 1$ the statement is obviously true, since each element of $\mathcal{J}$ has degree at least one. Thus we may assume that:

> *For every $r < i$ and every connection ω that allows a splitting at the region R_r with $\deg(m_r, k_r) < r$ there exist $j < r$ such that ω splits at R_j and $\deg(m_j, k_j) = j$.*

Consider now a connection ω that splits at R_i with $\deg(m_i, k_i) < i$. Then we can write ω as $[\omega_Y, \omega_N]$. The connection ω_N must allow splitting at R_{j_1} for some $i > j_1 \geq 1$, as otherwise ω_N could be written as $\omega_N = [\omega^i_{N,1}, \omega^i_{T_1}, \ldots, \omega^i_{T_i}]$, showing that $\deg(m_i, k_i) \geq i$ in contradiction to our assumption. Splitting ω_N at R_{j_1} we get: $\omega_N = [\omega_{T_1}, \omega_{N_1}]$ and the numbers(m_{j_1}, k_{j_1}) corresponding to that splitting. If $\deg(m_{j_1}, k_{j_1}) < j_1$, then we are done by an inductive assumption. If $\deg(m_{j_1}, k_{j_1}) = j_1$, then ω splits at R_{j_1} with the right equality $\deg(m_{j_1}, k_{j_1}) = j_1$. We claim that the last possibility: $\deg(m_{j_1}, k_{j_1}) > j_1$, implies that the connection ω_{T_1} splits at some other region R_{j_2} for $j_1 < j_2 < i$. If that claim is not true, then ω_{T_1} can be written as a composition of $i - j_1$ nontrivial connections on the cylinder between the regions R_{j_1} and R_i. Thus we have:

$$\deg(m_i, k_i) \geq \deg(m_{j_1}, k_{j_1})) + (i - j_1) > i$$

again contradicting our assumption. Thus we can write $\omega_{T_1} = [\omega_{T_2^+}, \omega_{T_2^-}]$. Define $\omega_{N_2} = [\omega_{N_1}, \omega_{T_2^-}]$. Same argument as above aplied for ω_{N_2} instead of ω_{N_1} shows that we either find the required splitting for ω, or we will find another splitting of $\omega_{T_2^+}$ at the region R_{j_3} for $j_2 < j_3 < i$. As there can be only i such splittings, this process must terminate by finding the R_j with the required properties.

The proof in the case when $deg(m_i, k_i) > i$ is the same as in the the previous case, exept that one has to reverse the "orientation" of the whole proof, i.e. start the induction from the point that is closest to the Y–side and replace all N's with Y's, and "+" with "−".

Let us now consider an element $(m,k) \in \mathcal{J}$ with maximal degree. There is a corresponding connection in $\mathcal{M}_k(N,m)$ with the splitting $[\omega_N, \omega_1, \ldots, \omega_{s-1}]$ that defines its degree. According to Corollary 3.11 the smallest dimension of a moduli space $\mathcal{M}_k(\mathbf{R} \times L, [i,j])$ including the gluing parameter from the N-side is no less than $4 \cdot (i-j)$. Thus

$$\begin{aligned} 2(d+n) &= \dim \mathcal{M}_{k_Y}(Y) + h_{m_Y} + \sum_{j=1}^{s-1} \dim \mathcal{M}^G_{k_j}(\mathbf{R} \times L, [m_j, m'_j]) + \\ &+ \dim \mathcal{M}_{k_N}(N, m_N) \geq 2d + h_X + 4(s-1) + 1 \geq 2d + 4s - 2 \end{aligned}$$

from which the last statement of the lemma follows. □

We conclude this section with one more comment on the sets $\mathcal{J}$. Let us assign to each edge of the graph $\mathcal{J}$ the minimal energy $\kappa = k/p$ that is required for the existence of ASD connection on the $\mathbf{Z}_p$ equivariant bundle with weights whose values are denoted at the ends this edge. This amount of energy has been established in Lemma 3.6. Then the energy of an arbitrary vertex can be obtained by summing the energies of the edges while going from the top of $\mathcal{J}$ to the given vertex. That is why on the graph representing $\mathcal{J}$ we can denote only the values of the equivalence classes of the flat connections $m \in \chi(L)$.

Example 3.19 When computing $D(z\sigma^n)$ for $\sigma \cdot \sigma = -6$ we obtain the following graph, that stabilizes:

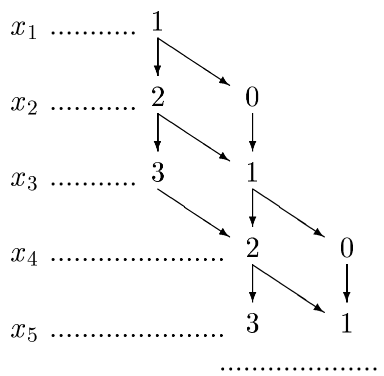

Figure 2. The graph $\mathcal{J}$ for $\sigma \cdot \sigma = -6$.

On this graph the points $x_1, x_2, \ldots$ denote the chosen points from the regions $R_1, R_2, \ldots$ All points on the graph lying on the level indicated by x_i correspond to open sets U_k^m that split at the region R_i. The above graph, according to Lemma 3.18, terminates at the level $[\frac{n+1}{2}]$. The meaning of the columns of this graph will be explained later.

4. COMPUTING INTERSECTION NUMBERS IN OPEN SETS U_k^m.

Lemma 4.1 *The intersection of the divisors $V_z \bigcap_{i=1}^{s} V_{\sigma_i}$ is empty in $U_{k'}^m$ unless the moduli space $\mathcal{M}_{k'}(N,m)$ contains a reducible connection.*

Before proceeding with the proof we want to comment on the terminology. The set U_k^m for a generic metric does not contain reducible connections, as $b_+(X) \geq 3$. Nonetheless *we shall informally say that A is a reducible connection in U_k^m* if the restriction of A to N is a reducible connection in $\mathcal{M}_k(N,m)$.
Proof: The open set $U_{k'}^m$ has dimension $2(d+n)$. On the other hand, this set viewed as a fibered product has the dimension

$$\dim\mathcal{M}_{k_0-k'}(Y,m) + h_m + \dim\mathcal{M}_{k'}(N,m)$$

As $b_+(X) = b_+(Y) \geq 3$, the intersection $\mathcal{M}_{k_0-k'}(Y,m) \cap V_z$ is transverse since it contains only irreducible connections. Thus $\dim\mathcal{M}_{k_1}(Y,m) \geq 2d$. If the divisors V_σ intersected along irreducible connection as well, then we would have: $2(d+n) \geq 2d + h_m + 2n$, in contradiction to the fact that $h_m = 1$ or 3. □

Because of this lemma we need to describe neighborhoods of reducible connections.

4.1. DESCRIPTION OF THE NEIGHBORHOODS OF THE REDUCIBLE CONNECTIONS.

Let us recall that our main purpose is to compute the intersection number

$$\mathcal{M}_k(X) \cap V_z \cap \bigcap_{i=1}^{n} V_{\sigma_i} \tag{4.2}$$

or dually, to evaluate the cohomology class $\mu(z\sigma^n)$ on the moduli space $\mathcal{M}_k(X)$. As we saw in the previous paragraph, this intersection is included in the set covered by the sum of open sets U_k^m and takes place near connections whose restriction to N is reducible. Thus our first task is to describe the neighborhoods of reducible connections in the moduli spaces $\mathcal{M}_{k_N}(N, m_i)$, and then we will describe a procedure for keeping the intersection (4.2) inside these neighborhoods. Before proceeding, we want to notice that there is one kind of noncompactness of the sets U_k^m that can be taken care of immediately. This noncompactness comes from the completely concentrated connections (so–called *bubbles*). Recall that the sets U_k^m are obtained via fibered products $\mathcal{M}_{k_Y}^x(Y,m) \underset{\partial}{\times} \mathcal{M}_k^x(N,m)$, where the point x belongs to the region R, on which the energy estimate excludes bubbling. Thus both basepoint fibrations extend to the $SO(3)$ bundles over the Uhlenbeck compactifications of these moduli spaces, and from this point on we will assume that the sets U_k^m contain completely concentrated connections.

When a reducible connection occurs in the top stratum, its neighborhood is modelled on $\mathbf{C}^n/S^1$. For the sake of later generalizations we set the following notation:

Definition 4.3
- Every neighborhood of a reducible connection is modelled on some stratified space divided by an S^1 action. Let K denote a compact neighborhood of the total space of this S^1 action. The space K/S^1 describes a compact neighborhood of

the reducible connection. In this case, $K = \{z \in \mathbf{C}^n | \|z\| \leq 1\}$; the general description of K is given below.

- Let $M = \partial K$.
- Let $r(K/S^1)$ be the image of K/S^1 under the deformation retraction shrinking the cone parameter to zero. (Thus in the case of a top stratum connection $r(K/S^1)$ is a point.) There is a corresponding deformation retraction of the set K, which we will denote by the same symbol. As S^1 acts trivially on $r(K)$, then we have $r(K/S^1) \cong r(K)$.

The set M/S^1 is often referred to as the *link* of the reducible connection. In this case $M/S^1 = \mathbf{C}P^{n-1}$. In the proof of Proposition (5.1.21) in [5] it is shown that the Pontrjagin class of the restriction of the adjoint bundle to the link of the reducible connection on the bundle $L \oplus \bar{L}$ is equal to

$$p_1(\mathbf{P}^{\mathrm{ad}}) = h^2 \times 1 + 4 \cdot h \times c_1(L) + 4 \cdot 1 \times c_1^2(L)$$

where h is the positive generator of $\mathrm{H}^2(\mathbf{C}P^{n-1})$. Thus we have the following:

Lemma 4.4 *On the link of a reducible connection on the top stratum in $\mathcal{M}_k(N,m)$ we have the following identities in $\mathrm{H}^2(M/S^1)$:*

$$\begin{aligned} \mu(\sigma) &= -\langle c_1(L), \sigma\rangle \cdot h \\ \mu(x) &= -\tfrac{1}{4} \cdot h^2 \end{aligned}$$

The second equation holds for points $x \in N$.

The description of reducible connections in the lower strata of $\mathcal{M}_k(N,m)$ has been worked out in [10]. For every sequence of integer numbers $n_1 \geq n_2 \geq \cdots \geq n_s$ there is a stratum St corresponding to s bubbles with multiplicities described by the numbers n_j. Let us denote by $k = k_i - \sum_{j=1}^{s} n_j$ and by $P = L \oplus \overline{L}$ the $SU(2)$ bundle with $c_2(P) = k$. Let FN be a principal $SO(4)$ bundle associated to the tangent bundle TN, and let us define $Fr = P \underset{N}{\times} FN$ to be a fibered product of the bundles P and FN over N. Then Fr is a principal $SO(3) \times SO(4)$ bundle over N. Following the notation from [10] and [13], for each $n > 0$ define $\bar{Z}_n$ to be the space of gauge equivalence classes of ideal ASD connections on S^4 with standard metric that satisfy:

1. They are based at the south pole.
2. They are centered at the north pole.
3. They are concentrated in the ϵ ball around the north pole.

The notions of "concentration of energy" and "ϵ ball" in the last statement should be understood in the following way: for a fixed annulus A contained in the complement of the ϵ ball, every ASD connection ω with "energy concentrated in the ϵ ball" can be glued along A to another ASD connection on the manifold N. The spaces $\bar{Z}_n$ have a natural $SO(3)$ action that changes the framing at the south pole, and the $SO(4)$ action that rotates the sphere. In addition to that there is an $\mathbf{R}_+$ action on each $\bar{Z}_n$ corresponding to conformal contraction toward north pole. Thus there exists a space Z_n such that $\bar{Z}_n$ is a cone cZ_n. For every $j = 1, 2, \ldots, s$ define $Gl_j = Fr \underset{SO(3)\times SO(4)}{\times} cZ_{n_j}$. As is proven in [13] the neighborhood K of the reducible connection in the strata St is

$$\left(\mathbf{C}^n \times \prod_{j=1}^{s} Gl_j\right)/S^1 \tag{4.5}$$

divided by the symmetry group permuting the Gl_j factors. The top stratum described above fits into the above description for $s = 0$. In Theorem 4.4.2 of [10] it has been shown that given a reducible connection ω, for $r = n_1 + n_2 + \cdots + n_s$, different strata of the form (4.5) can be glued together to form a smoothly stratified space $GP(\omega, r)$. This space allows the projection $\pi : GP(\omega, r) \to \Sigma^r(N)$ induced by an S^1 equivariant deformation retraction onto the r–fold symmetric product $\Sigma^r(N)$. The notation from Definition 4.3 extends to the case of a reducible connection in the lower strata. Lemma 4.7.4 in [10] gives a description of the class $\mu(\sigma)$ on the link of given reducible connection:

Theorem 4.6 *Let M be the link of a reducible connection $\omega \in GP(\omega, r)$. Then the class $\mu(\sigma) \in \mathrm{H}^2(M/S^1)$ is equal to*

$$\mu(\sigma) = \pi^*\Sigma^r(\sigma) - \langle c_1(L), \sigma\rangle \cdot c_M$$

where c_M is the first Chern class of the S^1 bundle $M \to M/S^1$, and $\Sigma^r(\sigma) \in \mathrm{H}^2(\Sigma^r(N))$ is the class induced from the Poincare dual to σ in $\mathrm{H}^2(N)$ by symmetrization.

Similarly we conclude that

$$-4\mu(x)|_{M/S^1} = -4\pi^*\Sigma^s(x) + c_M^2$$

Next we want to apply the gluing theorem to "attach" connections on the $Y-$ side to the neighborhoods of reducible connections on $N-$ side. Recall that the $SO(3)$ fibration $\mathcal{M}^x_{k_0-k}(Y, m)$ reduces to an S^1 fibration when m is nontrivial. In that case let $\mathcal{Q}$ denote this S^1–fibration. For the sake of uniform notation let $\mathcal{Q}$ be the whole total space of $\mathcal{M}^x_{k_0-k}(Y, m)$ when m is trivial. Notice that even in the later case we can still think of $\mathcal{Q}$ as of the total space of an S^1 fibration with the base $\mathcal{Q}/S^1$.

When the boundary value m is nontrivial, the basepoint fibration on the $Y-$side is simply $\mathcal{Q} \underset{S^1}{\times} SO(3)$ and the basepoint fibration $U^{m,x}_k \to U^m_k$ near a "reducible" connection can be described as the fibered product of:

$$\begin{tikzcd}
\mathcal{Q} \underset{S^1}{\times} SO(3) \arrow[dr] & & SO(3) \underset{S^1}{\times} K \arrow[dl] \\
& S^2 &
\end{tikzcd}$$

which gives the basepoint fibration $\mathcal{Q} \underset{S^1}{\times} (SO(3) \times K) \to \mathcal{Q} \underset{S^1}{\times} K$. This $SO(3)$ fibration reduces to an S^1 bundle with a total space $\mathcal{Q} \underset{S^1}{\times} (S^1 \times K) = \mathcal{Q} \times K$.

The case when m is trivial is even easier to describe. The $SO(3)$ fibration over U_k^m is:

$$\mathcal{Q} \times \left(SO(3) \underset{S^1}{\times} K \right) \to \mathcal{Q} \underset{S^1}{\times} K$$

which again admits a reduction whose total space is $\mathcal{Q} \times \left(S^1 \underset{S^1}{\times} K \right) = \mathcal{Q} \times K$.

4.2. DESINGULARIZATION OF $\mathcal{Q} \underset{S^1}{\times} K$.

To make the future computations easier we want to "enlarge" the set $\mathcal{Q} \underset{S^1}{\times} r(K) = \mathcal{Q}/S^1 \times r(K)$, onto which $\mathcal{Q} \underset{S^1}{\times} K$ deformation retracts. For any S^1 bundle $\mathcal{Q}$ let us denote by $\mathcal{Q}^C = \mathcal{Q} \underset{S^1}{\times} D^2$, the disc bundle associated to a circle bundle $\mathcal{Q}$. Consider a projection $\pi : \mathcal{Q} \underset{S^1}{\times} M^C \to \mathcal{Q} \underset{S^1}{\times} K$ that is the identity for every nonzero radius parameter of the disc D^2, and for the radius zero, $\pi([q],[m]) = ([q], r(m))$. Here $[.]$ denotes the equivalence class with the respect to the S^1 action. This map is well defined because of the S^1 equivariance of the retraction r. If α is any top dimensional cohomology class in $\mathrm{H}^{top}(\mathcal{Q} \underset{S^1}{\times} K, \mathcal{Q} \underset{S^1}{\times} M)$ then

$$\langle \alpha, [\mathcal{Q} \underset{S^1}{\times} K, \partial] \rangle = \langle \alpha, \pi_*([\mathcal{Q} \underset{S^1}{\times} M^C, \partial]) \rangle = \langle \pi^*\alpha, [\mathcal{Q} \underset{S^1}{\times} M^C, \partial] \rangle \tag{4.7}$$

Thus we can evaluate the pullback $\pi^*\alpha$ in the enlarged space $\mathcal{Q} \underset{S^1}{\times} M^C$, getting the same answer.

4.3. CLOSING NEIGHBORHOODS OF REDUCIBLE CONNECTIONS.

Next we wish to describe a procedure that will keep the intersection of divisors in (4.2) inside neighborhoods of reducible connections. Below, we describe compact spaces C containing neighborhoods $\mathcal{Q} \underset{S^1}{\times} M^C$ together with a technique for translating the intersection problem from U_k^m into the sum of these compact sets.

In case m is trivial, let $\mathcal{N}$ denote the based moduli space $\mathcal{M}^x_{k_N}(N, m)$ and $G = SO(3)$, and in the case when m is nontrivial, let $\mathcal{N}$ be the S^1 reduction of the above basepoint fibration and $G = S^1$. Then the set $U_k^m = \mathcal{Q} \underset{G}{\times} \mathcal{N}$.

Let $\mathcal{M}_k(N, m) = \mathcal{N}/G$ contain S^1 reducible connections $A_1, A_2, \ldots, A_s$, each contained in the compact neighborhood K_i/S^1. In the set U_k^m each A_i is contained in a neighborhood of the form $\mathcal{Q} \underset{S^1}{\times} K_i$. Let $q_i : \mathcal{Q} \underset{S^1}{\times} M_i \to \mathcal{Q}/S^1 \times M_i/S^1$ denote projections of the boundary of the above neighborhoods. The cohomology classes $\mu(\sigma)$ and $\mu(x)$ can be obtained from the pullback via q_i of the first Chern class of the bundle $M_i \to M_i/S^1$. Since the dimension of $\mathcal{Q}/S^1 \times M_i/S^1$ is 2 less than the dimension of U_k^m, the top dimensional cohomology class will evaluate trivially on it. Thus it is reasonable to compactify $\mathcal{Q} \underset{S^1}{\times} M_i^C$ as:

$$C_i = \mathcal{Q} \underset{S^1}{\times} M_i^C \underset{q_i | \mathcal{Q} \underset{S^1}{\times} M_i}{\cup} \mathcal{Q}/S^1 \times M_i/S^1 \tag{4.8}$$

The space that we added to $\mathcal{Q} \underset{S^1}{\times} M_i^C$ is homotopy equivalent to either of the two spaces: $\mathcal{Q}^C \underset{S^1}{\times} M_i$ or $\mathcal{Q} \underset{S^1}{\times} M_i^C$. Thus the projection may be replaced by an inclusion giving:

$$C_i \simeq \mathcal{Q} \underset{S^1}{\times} M_i^C \underset{\mathcal{Q} \underset{S^1}{\times} M_i}{\cup} \mathcal{Q}^C \underset{S^1}{\times} M_i \tag{4.9}$$

$$\simeq \mathcal{Q} \underset{S^1}{\times} M_i^C \underset{\mathcal{Q} \underset{S^1}{\times} M_i}{\cup} \mathcal{Q} \underset{S^1}{\times} M_i^C \tag{4.10}$$

To define the class $\mu(\sigma)$ we have to pull back the bundle $M_i \to M_i/S^1$ over the space $\mathcal{Q} \underset{S^1}{\times} M_i$, thus it is more convenient to use the construction (4.9). Then $\mu(\sigma)$ is a multiple of the first Chern class of the S^1 bundle:

$$\mathcal{Q} \times M_i^C \underset{\mathcal{Q} \times M_i}{\cup} \mathcal{Q}^C \times M_i \to \mathcal{Q} \underset{S^1}{\times} M_i^C \underset{\mathcal{Q} \underset{S^1}{\times} M_i}{\cup} \mathcal{Q}^C \underset{S^1}{\times} M_i$$

The total space of this bundle is the boundary of $\mathcal{Q}^C \times M_i^C$, which can be considered as the S^3 bundle $S\left(\mathcal{Q}^\mathcal{C} \times M_i^C\right)$ over $\mathcal{Q}/S^1 \times M_i/S^1$. Thus C_i is $P\left(\mathcal{Q}^C \times M^C\right)$, the projectivization of the bundle $\mathcal{Q}^C \times M_i^C$. The first Chern class of the bundle $S\left(\mathcal{Q}^\mathcal{C} \times M_i^C\right) \to P\left(\mathcal{Q}^\mathcal{C} \times M_i^C\right)$ is the class h that restricts as a generator of each fiber $\mathbf{C}P^1$.

The next two lemmas justify our choice of the above compactification:

Lemma 4.11 *Let w be a fixed divisor in $\mathcal{M}_k(N, m)$. Let $\mathcal{M}^*$ be the complement of the sum of the neighborhoods $\coprod_i K_i/S^1$. Then there is a one–to–one correspondence between divisors V on U_k^m that pull back from the divisor w, and collections of divisors $\hat{V}_i$ on C_i such that $\hat{V}_i \cap (\mathcal{Q}/S^1 \times M_i/S^1) = [\mathcal{Q}/S^1] \times [\hat{v}_i]$, and $(\hat{v}_1, \ldots, \hat{v}_s) = \partial[w]$. Here ∂ is the boundary map ∂ from the Mayer–Vietoris sequence:*

$$\cdots \to \mathrm{H}_*(\mathcal{M}^*) \oplus \bigoplus_i \mathrm{H}_*(K_i) \to \mathrm{H}_*(\mathcal{M}_k(N, m)) \xrightarrow{\partial} \bigoplus_i \mathrm{H}_{*-1}(M_i/S^1) \to \cdots$$

Proof: Given the divisor $V \subset U_k^m$ that pulls back from $w \subset \mathcal{M}_k(N,m)$ define $v_i = w|_{M_i/S^1}$. Using the deformation retractions $r_i : C_i \setminus \mathcal{Q} \underset{S^1}{\times} M_i^C \to \mathcal{Q}/S^1 \times M_i/S^1$, we can construct the divisors $\hat{V}_i = V|_{\mathcal{Q} \underset{S^1}{\times} M_i^C} \underset{\mathcal{Q} \underset{S^1}{\times} M_i}{\cup} r^{-1}([\mathcal{Q}/S^1] \times v_i)$.

Going in the other direction: given a collection of divisors $(\hat{V}_i, \hat{v}_i)$ define

$$V = \oplus_i \hat{V}|_{\mathcal{Q} \underset{S^1}{\times} M_i^C} \underset{\mathcal{Q} \underset{S^1}{\times} M_i}{\cup} p_N^{-1}(w|)$$

where $p_N : \mathcal{Q} \underset{G}{\times} \mathcal{N} \to \mathcal{N}/G$ is the projection, and $w|$ is a restriction of the class w onto the set $\mathcal{N}/G \setminus \left(\coprod_i p_N(\mathcal{Q} \underset{S^1}{\times} M_i^C)\right)$. It can be directly seen that the composition of these two assignments gives the identity. □

Lemma 4.12 *Let $w_1, \ldots, w_r$ be fixed divisors on $\mathcal{M}_k(N,m)$ and $V_1, \ldots, V_r$ denote pull-backs of w_i's to U_k^m. Assume also that the sum of codimensions of V_i's is equal to the dimension of U_k^m. Then the following signed intersections are equal:*

$$\bigcap_{i=1}^{r} V_i = \sum_{j=1}^{s} \bigcap_{i=1}^{r} \hat{V}_{i,j}$$

where $\hat{V}_{i,j}$ is the divisor on C_j corresponding to V_i as in Lemma 4.11.

Proof: Let $\hat{V}_{i,j}^t$ be a perturbation of $\hat{V}_{i,j}$ into transversal position. Since $\mathcal{Q}/S^1 \times M_j/S^1$ has codimension 2 in C_j, the one dimensional set $\{\bigcap_{i=1}^{r} \hat{V}_{i,j}^t \,|\, t \in [0,1]\}$ misses $\mathcal{Q}/S^1 \times M_j/S^1$, the set added in the compactification. As $\partial[w_i] = (\hat{v}_{i,1}, \ldots, \hat{v}_{i,s})$, and this is a homological condition, we can use Lemma 4.11 for all $\hat{V}_{i,j}^t$, to get a corresponding perturbation of the divisors $V_{i,j}^t$. Because the dimension of $\mathcal{M}_k(N,m)$ is lower than the dimension of U_k^m the perturbed divisors cannot intersect outside the set $\mathcal{Q} \underset{S^1}{\times} (\mathcal{N} \setminus (\coprod_i K_i))$. Thus the intersection number of V_i is equal to the intersection number of $\hat{V}_{i,j}$. □

The top stratum of the moduli space $\mathcal{M}_{k_N}(N,m)$ has a reducible connection on the bundle $\mathbf{L}$ if and only if $c_1^2(\mathbf{L}) = k_N$. Thus if the reducible connection exists, it is unique. Similarly on the stratum in which the background connections have energy $k_N - s$ there exists a unique reducible connection if and only if $c_1^2(\mathbf{L}') = k_N - s$. Thus every reducible connection $A_j \in \mathcal{M}_{k_N}(N,m)$ has uniquely assigned l_j, the first Chern class of the reduction of the bundle $\mathbf{L}_j$ and $s_j = k_N - c_1^2(\mathbf{L}_j)$. We can perform the closing–up construction for every link M_j of A_j separately, getting the following equivalent of Theorem 4.6 for the spaces C_j:

Corollary 4.13 *Under the above correspondence of cohomology classes we have the following equalities in* $\mathrm{H}^*(C_j)$*:*

$$\mu(\sigma) = \pi^* \Sigma^{s_j}(\sigma) - l_j \cdot h_j$$

where the class h_j restricts as the generator of each fiber $\mathbf{C}P^2 \hookrightarrow P(\mathcal{Q}^C \times M_j^C)$. Similarly

$$-4\mu(x_r) = \begin{cases} -4\pi^*\Sigma^{s_j}(x_r) + h_j^2 & \textit{if } r \leq i \\ c_Q^2 & \textit{if } r > i \end{cases}$$

We will need some properties of the characteristic classes of the bundle $P(\mathcal{Q}^C \times M^C)$ which we now evaluate. The Künneth formula provides an embedding

$$\mathrm{H}^*(M/S^1) \to \mathrm{H}^*(\mathcal{Q}/S^1 \times M/S^1)$$

that sends $x \in \mathrm{H}^*(M/S^1)$ to $1 \otimes x$. In the description that follows, by cohomology classes $x \in \mathrm{H}^*(M/S^1)$ or $y \in \mathrm{H}^*(\mathcal{Q}/S^1)$ we shall mean their images $1 \otimes x$ or $y \otimes 1$ in $\mathrm{H}^*(\mathcal{Q}/S^1 \times M/S^1)$. From the definition of characteristic classes we get the relation $h^2 = c_1(\mathcal{Q}^C \times M^C)h - c_2(\mathcal{Q}^C \times M^C)$. Notice that $c_1(\mathcal{Q}^C \times M^C) = c_M + c_Q$ and $c_2 = c_Q c_M$, which in vector notation $(a, b) = a \cdot h + b$ can be written as $h^2 = A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $A = \begin{pmatrix} c_M + c_Q & 1 \\ -c_Q c_M & 0 \end{pmatrix}$. By elementary computations we obtain the following lemma:

Lemma 4.14 *We have*

$$h^{n+2} = \frac{c_M^{n+2} - c_Q^{n+2}}{c_M - c_Q} \cdot h - \frac{c_M^{n+1} - c_Q^{n+1}}{c_M - c_Q} \cdot c_M c_Q$$

Proof: Using the expression for h^2 we get:

$$h \cdot (ah + b) = ah^2 + bh = [a(c_M + c_Q) + b] \cdot h \; - \; a \cdot c_M c_Q = A\begin{pmatrix} a \\ b \end{pmatrix}$$

Thus we have

$$h^{n+1} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Set

$$S = \begin{pmatrix} 1 & 1 \\ -c_M & -c_Q \end{pmatrix}$$

A direct check verifies the relation:

$$A = S \begin{pmatrix} c_Q & 0 \\ 0 & c_M \end{pmatrix} S^{-1}$$

thus

$$A^n = S \begin{pmatrix} c_Q^n & 0 \\ 0 & c_M^n \end{pmatrix} S^{-1}$$

from which the result follows. $\square$

The situation is different when the reducible background connection is trivial. In that case there is no direct relation between $\mu(\sigma)^2$ and $\mu(x)$. We need to supply a further discussion with some more notation. Notice first that whenever the moduli space $\mathcal{M}_k(N,m)$ admits a trivial background connection, then k must be an integer and $m=0$. Thus on the Y side we always have the $SO(3)$ fibraton $\mathcal{Q}=\mathcal{M}^x_{k_Y}(Y,0)$. Let M be the restriction of the basepoint fibration over the link of the trivial connection $M/SO(3)\subset\mathcal{M}_k(N,m)$ with the projection $\pi: M/SO(3)\to\Sigma^k(N)$.

We have the following analog of Theorem 4.6:

Lemma 4.15 *We have the following equalities of classes in* $\mathrm{H}^*(M/SO(3))$*:*

$$\mu(\sigma)=\pi^*\Sigma^s(\sigma)$$

$$\mu(x_j)=\pi^*\Sigma^s(x_j)-\tfrac{1}{4}P_M$$

where P_M is the first Pontrjagin class of the fibration $M\to M/SO(3)$.

For any $SO(3)$ fibration $\mathcal{Q}$ define $\mathcal{Q}^G=\mathcal{Q}\underset{SO(3)}{\times}cSO(3)$ and let p_Q denote its first Pontrjagin class. A similar construction to (4.9) produces an $SO(3)$ fibration:

$$\partial(\mathcal{Q}^G\times\mathcal{P}^G)\to\partial(\mathcal{Q}^G\underset{SO(3)}{\times}\mathcal{P}^G) \tag{4.16}$$

which we shall denote by $S_G(\mathcal{Q}^G\times\mathcal{P}^G)\to P_G(\mathcal{Q}^G\times\mathcal{P}^G)$, or even $S_G\to P_G$ when the two other $SO(3)$ bundles will be obvious from the context. Let H denote the Pontrjagin class of this fibration. As in the case of S^1 fibrations, there is a further projection $P_G(\mathcal{Q}^G\times M^G)\to\mathcal{Q}/G\times\mathcal{P}/G$, whose fiber is $\Sigma SO(3)$, the suspension of $SO(3)$. The fibration $S_G\to P_G$ restricts over each fiber of P_G as the fibration:

$$SO(3)*SO(3)\to\Sigma SO(3)$$

and the first Pontrjagin class is 2. As before, the computations of the intersections of the dual classes in the neighborhood of the trivial connection can be translated into the intersection of corresponding divisors in $P_G(\mathcal{Q}^G\times M^G)$. As in the case of S^1 reducible connections, there is a correspondence between the classes that pull back from the moduli space containing $M/SO(3)$ and the classes in the compactification (4.16). Using the above correspondence and Theorem 4.6 we get the following corollary:

Corollary 4.17 *Under the above correspondence of cohomology classes we have the following equalities:*

$$\mu(\sigma)=\pi^*\Sigma^s(\sigma) \tag{4.18}$$

where the class H restricts as twice generator of each fiber $P_G(\mathcal{Q}^G\times M^G)$. Similarly

$$-4\mu(x_j)=\begin{cases}-4\pi^*\Sigma^s(x_j)+H & \textit{if } j\le i\\ p_Q & \textit{if } j>i\end{cases}$$

It follows from the corollary that in this neighborhood, the top power of the classes $\mu(\sigma)$ evaluates trivially, so it seems that these neighborhoods are redundant. But our method of computing $\mu(\sigma)^{top}$ on the whole moduli space $\mathcal{M}_{k_X}(X)$ requires evaluating in $P_G(\mathcal{Q}^G \times M^G)$ terms of the form $\mu(\sigma)^\alpha \cdot \mu(x_1)^{\alpha_1} \cdots \mu(x_s)^{\alpha_s}$ for various $\alpha, \alpha_1, \ldots, \alpha_s$, which may give a nontrivial contribution in P_G. So let us state the following analog of Lemma 4.14:

Lemma 4.19 *With the notation above we have:*

$$H^{n+2} = \frac{p_P^{n+2} - p_Q^{n+2}}{p_M - p_Q} \cdot H - \frac{p_P^{n+1} - p_Q^{n+1}}{p_P - p_Q} \cdot p_P p_Q$$

Proof: As $S^1 \subset SO(3)$, then there is an S^1 fibration $\mathcal{Q} \to \mathcal{Q}/S^1$ with the first Chern class $c_Q \in \mathrm{H}^2(\mathcal{Q}/S^1)$. Let $p : \mathcal{Q}/S^1 \to \mathcal{Q}/SO(3)$ denote the projection of the fibration with the fiber S^2. Then since $p^*\left(\mathcal{Q} \underset{SO(3)}{\times} \mathbf{C}^3\right) = \mathcal{Q} \underset{S^1}{\times} \mathbf{C}^3$ we have:

$$(4.20)\quad p^*(-p_Q) = p^*\left(c_2(\mathcal{Q} \underset{SO(3)}{\times} \mathbf{C}^3)\right) = c_2\left(p^*(\mathcal{Q} \underset{SO(3)}{\times} \mathbf{C}^3)\right) = c_2(\mathcal{Q} \underset{S^1}{\times} \mathbf{C}^3) = -c_Q^2$$

Observe also that for every fiber $S^2 \subset \mathcal{Q}/S^1$, $c_2[S^2] = 2$. Thus from Leray–Hirsh theorem we get that the map $\langle \mathrm{H}^0(S^2) \oplus \mathrm{H}^2(S^2)\rangle \otimes \mathrm{H}^*(\mathcal{Q}/SO(3)) \to \mathrm{H}^*(\mathcal{Q}/S^1)$ sending $i^*(c_Q) \otimes \alpha \to p^*(\alpha) \cup c_Q$ is monomorphism, and in particular also p^* is monomorphism.

After these general remarks about $SO(3)$ fibrations let us come back to the construction described in the Lemma. Let h denote the first Chern class of the bundle $\partial(\mathcal{Q}^G \times \mathcal{P}^G) \to \partial(\mathcal{Q}^G \underset{S^1}{\times} \mathcal{P}^G)$. Then using Lemma 4.14 and (4.20) we have:

$$(4.21)\qquad p^*(H) = h^2 = (h(c_Q + c_P) - c_Q c_P)$$

Thus:

$$p^*(H^2) = h^4 = \frac{c_P^4 - c_Q^4}{c_P - c_Q} \cdot h - \frac{c_P^3 - c_Q^3}{c_P - c_Q} c_P c_Q =$$

$$\frac{c_P^4 - c_Q^4}{c_P^2 - c_Q^2} \cdot h(c_P + c_Q) - \frac{c_P^3 - c_Q^3}{c_P - c_Q} \cdot c_P c_Q$$

which by (4.21) can be written as:

$$\frac{c_P^4 - c_Q^4}{c_P^2 - c_Q^2} \cdot [p^* H + c_P c_Q] - \frac{c_P^3 - c_Q^3}{c_P - c_Q} \cdot c_P c_Q =$$

$$= p^*(H) \cdot (c_P^2 + c_Q^2) + (c_P c_Q) \cdot [-c_P c_Q] = p^*\left(H[p_P + p_Q] - p_P p_Q\right)$$

As p^* is a monomorphism, the above proves the same relation that we had in Lemma 4.14 with Chern classes replaced by Pontrjagin classes. Thus the result follows. □

Thus in this paragraph we have established the relations between the class $\mu(\sigma)$ and Pontrjagin classes of basepoint fibration in each of the sets U_k^m separately. The next section deals with the problem of matching these descriptions together.

4.4. COMPACTIFICATION OF THE U_k^m.

The results of the previous paragraph are based on the assumption that the spaces M/S^1 and $\mathcal{Q}/S^1$ are compact. However this assumption is not valid in general. Even after taking the Uhlenbeck compactification, the sets

$$U_k^m = \mathcal{M}_{k_Y}^x(Y) \underset{\partial}{\times} \mathcal{M}_k^x(N)$$

still remain open. This is because there are connections whose energy may escape through the cylindrical end. We want to describe the boundary of the sets U_k^m containing such connections. As we shall see in the next section, we need to understand the characteristic classes of the bundles over U_k^m that over the boundary ∂U_k^m pull back from a certain set $r(U)$ that has dimension at least 2 less than $\dim U_k^m$. Let us denote by $\bar{U}_k^m$ the compact space $U_k^m \underset{p}{\cup} r(U)$, where $p : \partial U_k^m \to r(U)$. Since there is an isomorphism of cohomology groups $\mathrm{H}^{top}(\bar{U}_k^m) \simeq \mathrm{H}^{top}(U_k^m, \partial)$, we shall refer to our construction as either compactifying the spaces U_k^m or as to splitting the global characteristic classes into the sum of relative ones.

We shall begin the description of the connections in the boundary of U_k^m with the generalization of the definition of based connections. Let us recall that if $\tilde{\mathcal{M}}_k(X)$ denotes the space of ASD connections in $\mathcal{A}$, then $\mathcal{M}_k^x(X) = \tilde{\mathcal{M}}_k(X) \underset{\mathcal{G}_E}{\times} E_x$. Thus we can similarly define a moduli space of connections based at two points as

$$\mathcal{M}_k^{x,y}(X) = \tilde{\mathcal{M}}_k(X) \underset{\mathcal{G}_E}{\times} (E_x \times E_y) \tag{4.22}$$

Note that $\mathcal{M}_k^{x,y}(X)$ can be also described as a fibered product $\mathcal{M}_k^x(X) \underset{\mathcal{M}_k(X)}{\times} \mathcal{M}_k^y(X)$. The projection $\mathcal{M}_k^{x,y}(X) \to \mathcal{M}_k^x(X)$ is in fact the quotient $\mathcal{M}_k^{x,y}(X)$ by $SO(3)_y$ acting on E_y. We shall use this definition for the moduli spaces of connections on the cylinders $\mathbf{R} \times L$ based at the different ends of $\mathbf{R}$. We indicate this by ${}^x\mathcal{M}_k^y(\mathbf{R} \times L)$. Similarly, for a finite interval $[a,b] \subset \mathbf{R}$ we can define the space ${}^x\mathcal{M}_k^y([a,b] \times L)$ to be the space of connections over the cylinder $[a-1, b+1] \times L$ having small energy on the subcylinders $[a-1,a] \times L$ and $[b, b+1] \times L$ containing the points x and y.

The noncompactness of U_k^m is a direct consequence of noncompactness of the spaces $\mathcal{M}_{k_Y}(Y)$ and $\mathcal{M}_k(N)$, so it is sufficient to describe the compactification of these factor spaces. Recall that in order to glue back these factor spaces, we need to work with spaces of connections having small energy on the fixed cylinder R. We shall assume this throughout this paragraph. Let us focus first on the case of compactifying the space $\mathcal{M}_{k_Y}(Y)$. The boundary component corresponding to the splitting at one region can be described as the fibered product:

$$\mathcal{M}_{k_Y}^y(Y, m_i) \underset{\partial}{\times} {}^y\mathcal{M}_{k_i}^x([a-1, b+1] \times L, [m_i, m]) / \, (SO(3)_y \times SO(3)_x) \tag{4.23}$$

The first Pontrjagin class of the $SO(3)_x$ action defines the point class $-4\mu(x)$. The total space of this $SO(3)_x$–fibration needs to be glued to the space $\mathcal{M}^x_{k_N}(N,m)$ to get the compactified set U^m_k. The time translation parameter of the space

$$\mathcal{M}_{k_i}([a-1,b+1]\times L,[m,m'])$$

is the collar parameter of the fibered space $\mathcal{M}_k(Y,m)$. Denote by $\mathcal{M}'_{k_i}([a-1,b+1]\times L) = \mathcal{M}_{k_i}([a-1,b+1]\times L)/\mathbf{R}$ and let

$$\begin{aligned} p:\mathcal{M}^y_{k_-}(Y,m_i)\underset{\partial}{\times}\mathcal{M}'^{,y}_{k_i}(T,[m_i,m])/SO(3)_y \to \\ \to \mathcal{M}_{k_-}(Y,m_i)\times\mathcal{M}'_{k_i}([a-1,b+1]\times L,[m_i,m]) \end{aligned}$$

be the projection. We define the compactification of $\mathcal{M}_{k_Y}(Y,m)$ as

$$\mathcal{M}_{k_Y}(Y,m)\underset{p}{\cup}\Big[\mathcal{M}_{k_-}(Y,m_i)\times\mathcal{M}'_{k_i}([a-1,b+1]\times L,[m_i,m])\Big]$$

Notice that the space that we have added in this compactification has codimension at least 2 in $\mathcal{M}^x_{k_Y}(Y,m)$. (One dimension is "lost" while dividing by time translations, and we also "lose" the gluing parameter between Y and the cylinder T). Thus we are in the situation described at the begining of this section, which allows us to define the compactified space $\mathcal{M}_{k_Y}(Y,m)$, or equivalently the relative class $\mathrm{H}^{top}(\mathcal{M}_{k_Y}(Y,m),\partial(\mathcal{M}_{k_Y}(Y,m))\ )$. We shall use the same symbol $\mathcal{M}_{k_Y}(Y,m)$ for the compactified space. We define the classes $\mu(x_i)$ on this compactification as pullbacks of the corresponding classes on the product $\mathcal{M}_{k_-}(Y,m_i)\times\mathcal{M}'_{k_i}([a-1,b+1]\times L,[m_i,m])$. Notice that with the class $\mu(y)$ for y in the splitting region, pulling back from $\mathcal{M}^y_{k_-}(Y,m_i)$ or $\mathcal{M}'^{,y}_{k_i}([a-1,b+1]\times L,[m_i,m])$ gives the same result.

While applying the same construction to the moduli space $\mathcal{M}_k(N,m)$ we get a similar splitting of that moduli space into factor spaces $\mathcal{M}^y_{k_N}(N,m')$ and the moduli space on the nearest cylinder $\mathcal{M}^y_{k_T}([a-1,b+1]\times L,[m',m])$. In this situation $\mathcal{M}^y_{k_N}(N,m')$ is not an $SO(3)$ fibration because of the presence of reducible connections. Thus we shall apply a construction similar to (4.9), relating the classes $\mu(\sigma)$ and $\mu(x_i)$ to the Pontrjagin class of the basepoint fibration based at x, which is at the end of N. This basepoint fibration based at x allows us to define the extension of the boundary value map $\partial:\mathcal{M}^x_{k_Y}(Y,m)\to\mathcal{R}(L)$, and thus the gluing of the compactified spaces $\mathcal{M}^x_{k_Y}(Y)$ and $\mathcal{M}^x_k(N)$. As a result we get a compactification of the set U^m_k. In general it may happen that the fibered product defining the boundary of $\mathcal{M}_k(Y)$ has a bigger number of factors. In this case we apply inductively the above procedure for constructing the compactification of an unbased moduli space. This construction is similar to the Floer compactification of the moduli spaces on the cylinder.

Now we want to ask whether the relations between $\mu(\sigma)$ and the point classes proven in Corollaries 4.13 and 4.18 hold on the sets C_i after compactifying the moduli spaces $\mathcal{M}_k(N,m)$ and $\mathcal{M}_{k_Y}(Y,m)$. With the definition of the compactified $\mu(\sigma)$ as above, the answer is positive only when there is only one reducible connection A_i is in the the set

$\mathcal{M}_{k_1}(N, m_1)$. (We used the relation between *this* connection and the Pontrjagin class of the basepoint fibration based at x to define the compactified class $\mu(\sigma)$.) Otherwise the cohomology class

$$\mu(\sigma) - \pi^* \Sigma^s(\sigma) - \langle c_1(\mathbf{L}) \ , \ \sigma \rangle \cdot h_\partial \tag{4.24}$$

whose evaluation inside the set C_i is zero, has nontrivial support in the neighborhoods of the other reducible connections A_j, $j \neq i$.

4.5. THE RELATIVE MAYER–VIETORIS SEQUENCE.

So far we have been dealing only with local computations in a single open set U_k^m coming from covering the part of the moduli space $\mathcal{M}_k(X)$ that contains the intersection of $V_z \cap \bigcap_{j=1}^{n} V_\sigma$. Now we want to show how to combine these local computations into a global one.

Assume first for simplicity that the whole moduli space M may be covered by just two open sets U_1 and U_2. As the moduli spaces of connections on bundles over $\mathbf{R} \times L$ allow time translations, then from (4.23) we see that $U_1 \cap U_2$ can be written as $(-2, 2) \times V'$, where V' is codimension 1 submanifold of M. Let us put $V_1 = U_1 \setminus (-1, 2) \times V'$ and $V_2 = U_2 \setminus (-2, 1) \times V'$ (thus V_i's are deformation retracts of U_i's, but they do not cover M). Then the cohomology group $\mathrm{H}^*(M)$ is a part of the relative Mayer–Vietoris sequence:

$$\begin{array}{ccccccc} \to & \mathrm{H}^*(M,\ V_1 \cup V_2) & \to \mathrm{H}^*(M,\ V_1) & \oplus & \mathrm{H}^*(M,\ V_2) & \to \mathrm{H}^*(M, \emptyset) & \to \\ & \| & \| & & \| & & \\ & \mathrm{H}^*(\Sigma V') & \mathrm{H}^*(U_2, \partial U_2) & & \mathrm{H}^*(U_1, \partial U_1) & & \end{array} \tag{4.25}$$

It follows from this sequence, that any top dimensional class in $\mathrm{H}^{top}(M)$ is equal to the sum of two relative classes in $\mathrm{H}^*(U_1, \partial U_1)$ and in $\mathrm{H}^*(U_2, \partial U_2)$ modulo a "correction term" from $\mathrm{H}^*(\Sigma V')$.

In general, when the set M is covered by open sets $U_1, U_2, \ldots, U_r$, let us denote the complements of these sets by $V_i = M \setminus U_i$. For any subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, r\}$ let $V_{i_1, i_2, \ldots, i_k} = V_{i_1} \cup \cdots \cup V_{i_k}$, and $V^{i_1, i_2, \ldots, i_k} = V_{i_1} \cap \cdots \cap V_{i_k}$. Then we have the following generalization of the relative Mayer–Vietoris sequence:

Theorem 4.26 *Let M be a manifold containing the submanifolds $V_1, \ldots, V_r$, not necessarily covering M. Then for every class $\mu \in \mathrm{H}^{top}(M, V_{1,2,\ldots,r})$ there exist classes $\mu_{i_1,\ldots,i_k} \in \mathrm{H}^{top}(M, V^{i_1,\ldots,i_k})$ for $\{i_1, i_2, \ldots i_k\} \subset \{1, 2, \ldots, r\}$, such that:*

$$\langle \mu \ , \ [M, V_{1,2,\ldots,r}] \rangle = \sum_k (-1)^{k+1} \sum_{\{i_1, i_2, \ldots i_k\}} \langle \mu_{i_1, i_2, \ldots i_k} \ , \ [M, V^{i_1, i_2, \ldots i_k}] \rangle$$

Proof: We shall prove this theorem by induction on the number of sets V_i. For $r = 2$ the theorem follows from the relative Mayer–Vietoris Theorem described above. Assume thus that this theorem is valid for all $n < r$. The sequence (4.25) for $M_1 = V_1$ and $M_2 = V_{2,3,\ldots,r}$ is:

$$\mathrm{H}^{top}(M, V_1 \cup V_{2,3,\ldots,r}) \to \mathrm{H}^{top}(M, V_1) \oplus \mathrm{H}^{top}(M, V_{2,3,\ldots,r}) \to \mathrm{H}^{top}(M, V_{1,2,\ldots,r}) \to 0 \tag{4.27}$$

Hence, using the inductive assumption, there exist classes $\mu_{i_1,\ldots,i_s} \in \mathrm{H}^{top}(M, V_1 \cup V_{2,3,\ldots,r})$, where $\{i_1, \ldots, i_s\} \subset \{2, 3, \ldots, r\}$, a class $\mu_1 \in \mathrm{H}(M, V_1)$, and a class $\alpha \in \mathrm{H}^{top}(M, V_1 \cup V_{2,3,\ldots,r})$ such that

$$\mu = \mu_1 + \sum_k (-1)^{k+1} \sum_{\substack{\{i_1,i_2,\ldots i_k\}\subset \\ \subset\{2,3,\ldots,r\}}} \mu_{i_1,i_2,\ldots i_k} - \alpha$$

By excision we have: $\mathrm{H}^*(M, V_1 \cup V_{2,3,\ldots,r}) \simeq \mathrm{H}^*(U_1, \overline{V}_{2,3,\ldots,r})$, where $U_1 = M \setminus V_1$, and $\overline{V}_i = U_1 \cap V_i$. We shall use the same symbol $\alpha \in \mathrm{H}^*(U_1, \overline{V}_{2,3,\ldots,r})$ for the image of $\alpha \in \mathrm{H}^*(M, V_1 \cup V_{2,3,\ldots,r})$ under the excision isomorphism. Using again the inductive assumption for the set U_1 containing sets $\overline{V}_i$ we get

$$\alpha = \sum_k (-1)^{k+1} \sum_{\substack{\{i_1,i_2,\ldots i_k\}\subset \\ \subset\{2,3,\ldots,r\}}} \alpha_{i_1,i_2,\ldots i_k}$$

where $\alpha_{i_1,i_2,\ldots i_k} \in \mathrm{H}^{top}(U_1, \overline{V})$. By excision $\mathrm{H}^{top}(U_1, \overline{V}) \simeq \mathrm{H}^{top}(M, V^{1,i_1,i_2,\ldots,i_k})$, thus $\alpha_{i_1,i_2,\ldots i_k}$ correspond to $\mu_{1,i_1,i_2,\ldots i_k}$, which concludes the proof. $\square$

Corollary 4.28 *When the sets $U_1, U_2, \ldots, U_r$ cover the space M, then every class $\mu \in \mathrm{H}^{top}(M)$ is equal to a combination of the classes $\mu_{i_1,i_2,\ldots,i_k} \in \mathrm{H}^{top}(\overline{U}_{i_1,i_2,\ldots,i_k})$, where $\overline{U}_{i_1,i_2,\ldots,i_k}$ is a compactification of the intersection $U_{i_1,i_2,\ldots,i_k}$ described in previous paragraph.*

Proof: When the sets $\{U_i\}_{i=1}^r$ cover M, then $V_{1,2,\ldots,r} = \emptyset$. Thus from Theorem 4.26 it is sufficient to prove that $\mathrm{H}^{top}(M, V^{i_1,i_2,\ldots,i_k}) \simeq \mathrm{H}^{top}(\overline{U}_{i_1,i_2,\ldots,i_k})$. The excision isomorphism gives us $\mathrm{H}^*(M, V^{i_1,i_2,\ldots,i_k}) \simeq \mathrm{H}^*(U_{i_1,i_2,\ldots,i_k}, \partial)$. Recall that the compactified sets $\overline{U}$ are of the form $U \underset{\phi}{\cup} K$, where K is a set of codimension at least 2. Let us denote $C_K = \partial U \underset{\phi}{\cup} K$. Then we have another excision isomorphism: $\mathrm{H}^*(U_{i_1,i_2,\ldots,i_k}, \partial) \simeq \mathrm{H}^*(\overline{U}, C_K)$. As the set C_K deformation retracts onto K, which is of codimension at least 2, then we have the last isomophism $\mathrm{H}^{top}(\overline{U}, C_K) \simeq \mathrm{H}^{top}(\overline{U})$ from the exact sequence of the pair $(\overline{U}, C_K)$. $\square$

Let Φ denote the set of these classes $\phi \in \mathrm{H}^{top}(\mathcal{M}_k(X))$ that are equal to a linear combination of terms of the form $a_{\alpha,\alpha_1,\ldots,\alpha_s}\mu(\sigma)^\alpha \cdot \mu(x_1)^{\alpha_1} \cdots \mu(x_s)^{\alpha_s}$. The specific compactification of the classes $\mu(\sigma)$ and $\mu(x_i)$ that we defined before allows us for every $\phi \in \Phi$ to define a "restriction map" $r_{i_1,i_2,\ldots,i_k} : \mathrm{H}^{top}(\mathcal{M}_k(X)) \to \mathrm{H}^{top}(\overline{U}_{i_1,i_2,\ldots,i_k})$ such that ϕ is a combination of the classes $r_{i_1,i_2,\ldots,i_k}(\phi)$ in the sense of Lemma 4.28. The following lemma will allow us to avoid dealing with the classes $r_{i_1,i_2,\ldots,i_k}(\phi)$ for $k > 1$:

Lemma 4.29 *Let M and $V_1, \ldots, V_r$ be as in Theorem 4.26. Let $\mu \in \mathrm{H}^{top}(M)$ be a cohomology class such that $\mu_i = 0 \in \mathrm{H}^{top}(M, V_i)$ for every $i = 1, 2, \ldots, r$. Then $\mu = 0 \in \mathrm{H}^{top}(M, V_{1,2,\ldots,r})$.*

Proof: This is proved by an induction on the number of sets V_i as in (4.26). For $r = 2$ the statement follows from the relative Mayer–Vietoris sequence. The inductive assumption implies that $\mu \in \mathrm{H}^{top}(M, V_{2,3,\ldots,r})$ is zero. Thus the sequence (4.27) gives us the lemma. $\square$

5. THE PROOF OF THEOREM 1.3.

The proof of Theorem 1.3 is based on the idea of replacing the classes $\mu(\sigma)$ with some appropriate point classes $\mu(x_i)$ by using relations described in Lemmas 4.13 and 4.17. In other words we would like to find a linear combination of the form:

$$\mu(\sigma)^n + \sum_{\alpha<n} a_{\alpha,\alpha_1,\ldots,\alpha_s}\mu(\sigma)^\alpha \cdot \mu(x_1)^{\alpha_1}\cdots\mu(x_s)^{\alpha_s} \tag{5.1}$$

that would evaluate as zero on every set U_i. In the above formula, as well as in later notation, we shall often skip the factor $\mu(z)$ that appears in every term. Then from Corollary 4.28 it follows that (5.1) is zero on the whole of $\mathcal{M}_k(X)$, thus giving the main result.

We shall begin by defining a form that will evaluate trivially on a part of the diagram $\mathcal{J}$ defined in (3.16). Let $\{U_i\}_{i=1}^s$ be the open sets of the cover corresponding to the first column of $\mathcal{J}$.

Theorem 5.2 *Let $\sigma\cdot\sigma = -p$, and let s be the number of elements in $\chi(L(p,1)\ )$. Let $n > 2s$, and set $r = \frac{n-2s}{2}$ when n is even, and $r = \frac{n-2s+1}{2}$ when n is odd. There is an $a = (a_1, a_2, \ldots, a_s)$ such that $\phi(a)$ defined by:*

$$\phi(a) = \mu(\sigma)^n - a_s\mu(\sigma)^{2s}\mu(x_s)^r + a_{s-1}\mu(\sigma)^{2s-2}\mu(x_{s-1})^{r+1} + \cdots + a_1\mu(\sigma)^2\mu(x_1)^{r+s-1}$$

when n is even, and

$$\phi(a) = \mu(\sigma)^n - a_s\mu(\sigma)^{2s-1}\mu(x_s)^r + a_{s-1}\mu(\sigma)^{2s-3}\mu(x_{s-1})^{r+1} + \cdots + a_1\mu(\sigma)^1\mu(x_1)^{r+s-1}$$

when n is odd, evaluates trivially on the set $\bigcup_{1\le i\le s} U_i$.

Proof: We shall give the proof in the case when n is even. The proof in the other case is analogous. For convenience let us set $p(x_i) = -4\mu(x_i)$. Using Corollary 4.13 for the set U_s we get the relation: $\mu(\sigma)^2 = s^2\cdot p(x_i)$ for $i \le s$. Thus, in U_s we get:

$$\phi(a) = \left(s^{2r} + a_s + \frac{1}{s^2}a_{s-1} + \cdots + \frac{1}{(s^2)^{s-1}}a_1\right)\mu(\sigma)^{2s}p(x_s)^r$$

While considering the other sets U_i we have to use the following lemma, which is a direct consequence of Lemma 4.14:

Lemma 5.3 *Let $i > k$, and let the numbers α, β, and γ be such that $\gamma+\alpha \ge \dim\mathcal{M}_{\frac{k^2}{p}}(N,k)$. Then in the set U_k we have:*

$$\mu(\sigma)^\gamma\mu(x_k)^\alpha\mu(x_i)^\beta = \mu(\sigma)^\gamma\mu(x_k)^{\alpha+\beta}$$

From this Lemma and the relation $\mu(\sigma)^2 = k^2p(x_i)$ for $i \le k$ we get in U_k:

$$\phi(a) \;=\; \mu(\sigma)^{2k}\Big((k^2)^{r+s-k}p(x_k)^{r+s-k} + (k^2)^{s-k}a_s\cdot p(x_k)^{s-k}p(x_s)^r + \cdots$$

$$+ \quad \frac{1}{(k^2)^{k-1}} p(x_k)^{r+s-k} a_1\Big)$$
$$= \quad \mu(\sigma)^{2k} p(x_k)^{r+s-k} \left((k^2)^{r+s-k} + (k^2)^{s-k} a_s + \cdots + \frac{1}{(k^2)^{k-1}} a_1 \right)$$

Thus we get a system of s equations with s variables, whose matrix is the Vandermonde matrix:

$$W_s = \begin{pmatrix} (s^2)^{s-1} & \ldots & (s^2) & 1 \\ \cdots & \cdots & \cdots & \cdots \\ (2^2)^{s-1} & \ldots & (2^2) & 1 \\ 1 & \ldots & 1 & 1 \end{pmatrix} \tag{5.4}$$

whose determinant is $\prod_{1 \le i < j \le s} (j^2 - i^2)$. In particular, this determinant is nonzero, thus we can always find the numbers $a_1, \ldots, a_s$. $\square$

In order to remove the assumption $n > 2s$ from the above theorem we need to define the relative invariants introduced in [8].

Definition 5.5 Let Y be a 4–manifold with a cylindrical end $\mathbf{R} \times L$, and $\mathcal{M}_{k_Y}(Y, m)$ a compactified moduli space. Assume that m is nontrivial, and that c denotes the first Chern class of the S^1 fibration $\partial^{-1}(pt)$. Then define the relative Donaldson invariant $D_Y[m] : \mathbf{A}(Y) \to \mathbf{R}$ by:

$$D_Y[m](z \cdot c^\beta) = \langle \mu(z) \cdot c^\beta \ , \ \mathcal{M}_{k_Y}(Y, m) \rangle$$

In the situation when a closed manifold $X = Y \underset{L}{\cup} N$ we can assign a meaning to the relative invariants $D_Y[m]$ also in the case of trivial m. When $m = \frac{p}{2}$ we set $D_Y[m](z) = D_{X,\sigma}(z)$ to be a twisted $SO(3)$ invariant. When $m = 0$ we set $D_Y[m](z) = D_X(z)$, the usual invariant.

The class w_2 of that twisted invariant must evaluate as zero on Y and on N, thus it is a unique $\mathbf{Z}_2$ class that comes from $\mathrm{H}^1(L)$ in the cohomology sequence:

$$\to \mathrm{H}^1(L) \to \mathrm{H}^2(X) \to \mathrm{H}^2(Y) \oplus \mathrm{H}^2(N) \to$$

Since the $\mathbf{Z}_2$ reduction of the class σ satisfies these conditions, we have $w_2 = \sigma$.

Lemma 5.6 *Let $n = 2k \le 2s$ (where s is the number of elements in $\chi(L)$). Then there exist coefficients $a_1, \ldots, a_{k-1}$ such that the form*

$$\phi_k = \mu(\sigma)^{2k} - a_{k-1} \mu(\sigma)^{2k-2} p(x_{k-1}) - \cdots - a_1 \mu(\sigma)^2 p(x_1)^{k-1}$$

evaluates on the sum of the sets $\cup_{i=1}^k U_i$ as $(2k)! D_Y[k](z)$ in the case when k is a trivial element of $\chi(L)$, and evaluates as $\frac{(2k)!}{2} D_Y[k](z \cdot c)$ otherwise.

Proof: Let $W(k-1)$ denote the $k \times (k-1)$ matrix

$$\begin{pmatrix} -((k-1)^2)^{k-1} & ((k-1)^2)^{k-2} & \ldots & (k-1)^2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 1 & 1 & \ldots & 1 \end{pmatrix}$$

and let $W(k-1)^j$ denote the $(k-1)\times(k-1)$ matrix obtained from $W(k-1)$ by deleting j^{th} column from the right. Then according to Theorem 5.2 for $a_j = \frac{\det W(k-1)^j}{\det W_{k-1}}$ the form ϕ_k evaluates trivially on the sets U_i for $1 \le i < k$. So it remains to evaluate ϕ_k on the set U_k:

$$\begin{aligned}
\phi_k|_{U_k} &= -(k^2)^k + a_{k-1}(k^2)^{k-1} + \cdots + a_1 k^2 = \\
&= \frac{k^2}{\det W_{k-1}}\Big[-(k^2)^{k-1}\cdot \det W_{k-1} + a_{k-1}(k^2)^{k-2}\det W(k-1)^{k-1} + \cdots \\
&+ a_1 \det W(k-1)^1\Big] = \\
&= k^2\frac{\det W_k}{\det W_{k-1}} = \prod_{0\le i\le k-1}(k^2-i^2) = \frac{(2k)!}{2} \qquad (5.7)
\end{aligned}$$

When k is a trivial element of $\chi(L)$ and U_k corresponds to the set with the biggest amount of energy on the N–side that covers the intersection of divisors V_σ, then the bundle Q apearing in the Lemma 4.14 is $SO(3) \to S^2 = SO(3)/S^1$. As the first Chern class of this bundle is equal to 2, then we have an additional factor of 2 in the formula for $\phi_k|_{U_k}$ in this case. □

To avoid dealing with separate statements for two different cases $n > 2s$ and $n \le 2s$ we shall introduce the following notation:

Definition 5.8 Define the Donaldson invariants needed for evaluation $D(z\sigma^n)$:

$$D_\perp(z\sigma^{2m}x_{m+2d}^{k-2d-m}) = \begin{cases} D(z\sigma^{2m}x_{m+2d}^{k-2d-m}) & \text{if } k-2d-m>0. \\ D[m](c\cdot z) & \text{if } k-2d-m=0. \\ 0 & \text{if } k-2d-m<0. \end{cases}$$

when $n = 2k$, and

$$D_\perp(z\sigma^{2m-1}x_{m+2d}^{k-2d-m}) = \begin{cases} D(z\sigma^{2m-1}x_{m+2d}^{k-2d-m}) & \text{if } k-2d-m>0. \\ D[m](z) & \text{if } k-2d-m=0. \\ 0 & \text{if } k-2d-m<0. \end{cases}$$

in the case $n = 2k+1$.

The particular subscript of x_{m+2d} in the above formulas is taken from the fact that in the $(d+1)^{\text{st}}$ column of the diagram $\mathcal{J}$ the element with boundary value m lies in the level x_{m+2d}. Now we are ready to prove the theorem, from which (1.3) will follow:

Theorem 5.9 *Let X be a smooth manifold containing an embedded sphere S^2 with self-intersection $\sigma^2 = -p$. Let s be the number of elements in $\chi(L(p,1))$, and let $z \in \mathbf{A}(X)$ be a cohomology class such that $z\cdot\sigma = 0$. Then there exist constants a_j^r such that:*

$$D(\sigma^{2k}z) = \sum_{m=1}^{s}\sum_{d=0} a_m^d D_\perp(\sigma^{2m}x^{k-2d-m}z) \qquad (5.10)$$

when $n = 2k$, *and:*

$$D(\sigma^{2k+1} z) = \sum_{m=1}^{s} \sum_{d=0} a_m^d D_\perp(\sigma^{2m-1} x^{k-2d-m} z) \tag{5.11}$$

when $n = 2k + 1$.

Proof: We first concentrate on the case when $n = 2k$. Let $U_1, U_2, \ldots, U_s$ be the open sets that correspond to the first column of the diagram $\mathcal{J}$ (if there are fewer than s sets in the first column, thanks to the definition of $D_\perp$ we can assume that we completed that list by empty sets). Theorem 5.2 and Lemma 5.6 imply that there are constants $a_1^0, a_2^0, \ldots, a_s^0$ such that

$$\phi^0 = \sum_{m=1}^{s} a_m^0 D_\perp(z\sigma^{2m} x^{k-m})$$

evaluates trivially on the first column of $\mathcal{J}$. Let $\mathcal{J}^{(1)}$ be a diagram obtained from $\mathcal{J}$ by erasing its first column. For every vertex of $\mathcal{J}^{(1)}$, like in (3.16) we can assign an open set in $\mathcal{M}_{k-1}(X)$ (the moduli space of connections on X, with *one charge less* than the space that we started with.) Repeating the argument from Theorem 5.2 we can show that there exists a polynomial

$$\begin{aligned} \phi^1 &= a_s^1 \mu(\sigma)^{2s} \mu(x_{s+2})^{r-1} + a_{s-1}^1 \mu(\sigma)^{2s-2} \mu(x_{s+1})^r + \cdots + a_1^1 \mu(\sigma)^2 \mu(x_3)^{r+s-2} \\ &+ a_0^1 \mu(x_2)^{r+s-1} \end{aligned}$$

such that $\phi^0 - \phi^1$ evaluates trivially on the first column of $\mathcal{J}^{(1)}$. The difference $\phi^0 - \phi^1$ should be understood in the following way: as the diagram $\mathcal{J}^{(1)}$ has been obtained from $\mathcal{J}$ by erasing some of its elements, then we can say that there is an embedding $\mathcal{J}^{(1)} \hookrightarrow \mathcal{J}$. Then ϕ^1 we evaluate on the open set corresponding to a vertex of $\mathcal{J}^{(1)}$, whereas the ϕ^0 we evaluate on the corresponding set of $\mathcal{J}$. Repeating Theorem 5.2 for consecutive diagrams $\mathcal{J}^{(j)}$ we prove the theorem. $\square$

Proof of Theorem 1.3: Lemma 5.6 with Theorem 5.9 implies that the relative Donaldson invariants $D_Y[k]$ can be expressed in terms of $D(\sigma^{2k})$ and $D(\sigma^{2j} x^{k-j})$ for $j < k$. When $p = 2k$ then the character variety is $\chi(L) = \{0, 1, \ldots, k\}$. Since $\dim \mathcal{M}_\kappa(N, k) \geq 4k - 3$ and k is a trivial element of this character variety, then we need at least $2k$ classes σ to be able to express $D_Y[k]$ in the terms of invariants on X. When computing $D(\sigma^{\mathrm{odd}}$ only odd powers over σ can occur, that is why we can replace all relative invariants in 5.10 begining from $n = 2k + 2 = p + 2$. When $p = 2k + 1$, then the corresponding character variety is the same, but this time $k \in \chi(L)$ is a nontrivial element. That is why we can "reach" this k by $2k - 1$ classes σ, thus we are able to replace all relative invariants begining from $n = 2k + 1 = p$. $\square$

6. EXAMPLES OF COMPUTATIONS.

The theory that we have developed gives more than just Theorem 1.3. By using the results of this thesis we can compute the actual coefficients in some of the formulas given by (1.3). Below we give some examples of such computations.

Example 6.1 Let σ be the cohomology class of an embedded 2–sphere with self–intersection -4. Although Theorem 1.3 does not cover the case of $D(z\sigma^4)$, the only relative Donaldson invariant that appears in Theorem 5.9 is a twisted $SO(3)$ invariant with $w_2 = \sigma$, that is why we get the formula of the form:

$$D(z\sigma^4) = A \cdot D(z) + B \cdot D(z\sigma^2 x) + C \cdot D_\sigma(z)$$

We have already seen that the diagram $\mathcal{J}$ in this situation is

$$\begin{tikzcd} 1 \arrow[d] \arrow[dr] & \\ 2 & 0 \end{tikzcd}$$

and let us denote the corresponding open sets by $U^1_{1/4}$, U^2_1 and U^0_1. Then from Lemma 5.6 we have that $\phi^0 = \mu(\sigma)^4 - \mu(\sigma)^2 p(x_1)$ evaluates trivially on $U^1_{1/4}$. Since the coefficient B is obtained from the term $\mu(\sigma)^2 p(x_1) = -4\mu(\sigma)^2\mu(x_1)$, then we have $B = -4$. The coefficient C we obtain by evaluating ϕ^0 on U^2_1. The $\dim \mathcal{M}_1(N, 2) = 5$, thus the neighborhood of the reducible connection is modelled on $\mathbf{C}^3/S^1 = \mathbf{C}P^2$. From Lemma 4.13 it follows that $U^2_1 = P(SO(3)^C \times \gamma^C_2)$, where $\gamma_n \to \mathbf{C}P^n$ is a line bundle with $c_1(\gamma_n) = [\mathbf{C}P^1]$. If h is the generator of the fiber $\mathbf{C}P^1$ of P, then we have:

$$\mu(\sigma) = 2h; \qquad\qquad p(x_1) = h^2$$

Thus in U^2_1

$$\mu(\sigma)^4 - \mu(\sigma)^2 p(x_1) = \pm c([SO(3)/S^1]) \cdot (2^4 - 2^2) \cdot D_\sigma(z) = \pm 24 \cdot D_\sigma(z)$$

Here $c = 2[S^2]$ is the first Chern class of the fibration $SO(3) \to SO(3)/S^1$.

In order to determine the sign in the above equation we have to refer to the orientations of moduli spaces defined in [3]. By using a complex structure on the elliptic complex

$$\Omega^0(TX \otimes L) \to \Omega^1(TX \otimes L) \to \Omega^2_+(TX \otimes L)$$

where $L \to X$ is a complex line bundle, one can assign an orientation $o(L)$ to a reducible connection respecting the splitting $L \oplus \bar{L}$ in the case of $SU(2)$ connections, or respecting the splitting $L \oplus \mathbf{R}$ in the case of $SO(3)$ connection. Example 4.3 from [3] worked out for an arbitrary reducible bundle $L \oplus \bar{L}$ gives:

Claim 6.2 *The orientation $o(L)$ of a moduli space in the neighborhood of reducible connection on a bundle $E = L \oplus \bar{L}$ is $-\bar{n}\wedge$* (standard orientation on $\mathbf{C}P^n$)*, where $\bar{n}$ is the normal pointing away from the reduction. This orientation compares to the standard one with the sign $(-1)^{c_1(L)^2}$.*

For connections over the space N we have to change the sign in the above Claim, due to the fact that N has the orientation *opposite* to the complex one. The decomposition $X = Y\#N$ yields a splitting of arbitrary line bundle L into $L \cong L_Y\#L_N$, where $c_1(L) = c_1(L_N)$ on N. The neighborhood of the reducible connection on $L_N \oplus \overline{L}_N$, described as $SO(3) \underset{S^1}{\times} \mathbf{C}^n$, has the orientation $o(L)$. An arbitrary connection may be related to a reducible one through additions and subtractions of instantons. The set U_1^2 is a neighborhood of the reducible connection on $L_N \oplus \overline{L}_N$ with $c_1(L_N) = 2[S^2] \in \mathrm{H}^2(N)$. The associated $SO(3)$ connection is defined on $L_N^2 \oplus \mathbf{R}$. Thus as a bundle L on the whole X we can take the line bundle with $c_1(L) = -PD[S^2] = -\sigma$, since $-\sigma[S^2] = c_1(L_N^2) = 4$. Thus the complex orientation on U_1^2 compares with the standard one with the sign $(-1)^{\sigma^2} = (-1)^4$, giving the "+" sign at $D_\sigma(z)$.

There are no S^1 reducible connections inside the set U_1^0 but there is a trivial one. Following the discussion in [21], the restriction of the base–point fibration to the link of the trivial connection in $\mathcal{M}_1(N,0)$ is $Fr_+(N) \to N$, where $Fr_+(N)$ is the associated $SO(3)$–principal bundle of $\Lambda_+^2 TN$. Then $U_1^0 = P_G(SO(3)^G \times Fr_+(N)^G)$, and following Corollary 4.17 we have in this set:

$$\mu(\sigma) = \pi^*(\sigma); \qquad\qquad p(x_1) = -4\pi^*(x_1) + H$$

where H is a generator of the fiber $\Sigma SO(3)$ of $P_G(SO(3)^G \times Fr_+(N)^G)$. Thus

$$\phi^0 = -H[\Sigma SO(3)] \cdot \sigma^2[N] = -2 \cdot (-4) = 8$$

The additional "−" sign has to be added again due to the anti–complex orientation of N. Putting all three terms together we get

Theorem 6.3 *When σ is a class of S^2 with self-intersection -4, then the following formula holds:*

$$D(z\sigma^4) = -8 \cdot D(z) + 24 \cdot D_\sigma(z) - 4 \cdot D(z\sigma^2 x) \tag{6.4}$$

6.1. SECOND STRATUM CONNECTIONS OVER $\overline{\mathbf{C}P^2}$.

The next example requires the description of the reducible connections in the second stratum of the moduli space. The following presentation is based on [21, 17]. Let ω be a reducible connection on the bundle $L \oplus \bar{L}$ over X with $c_L = c_1(L)$. Assume that n is such that the neighborhood of $\omega \in \mathcal{M}_{c_L^2}(X)$ is modeled on $\mathbf{C}^n/S^1$. Then an open neighborhood of $\{\omega\} \times X$ in the compactified moduli space $\mathcal{M}_{c_L^2+1}(X)$ is homeomorphic to the bundle:

$$\mathbf{C}^n \underset{S^1_{\mathrm{ST}}}{\times} \left(P_L \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3)\,) \Big/ U(1) \right) \tag{6.5}$$

By the symbol $P \underset{X}{\times} Q$ we have denoted the fibered product of two fibrations P and Q over X. (the pullback of Q over P.) Let P_L denote the principal S^1 bundle associated to L, on which $U(1)$ acts in a natural way from the right. The $U(1)$ action on $cSO(3)$ is a *square* of the right action of $S^1 \hookrightarrow SO(3)$ via the map

$$\exp(i\theta) \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

For any complex line bundle L we have $L \underset{\tilde{S^1}}{\times} SO(3) \cong L^2 \underset{S^1}{\times} SO(3)$, where the $\tilde{S^1}$ action on $SO(3)$ is the square of the S^1 action. Thus in what will follow we shall replace the fibration P_L with P_{L^2} getting the standard $U(1) = S^1$ actions on all bundles appearing in the formula. The stabilizer S^1_{ST} acts in the standard way on $\mathbf{C}^n$ and on the left on the space $P_L \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3)) \Big/ U(1)$. Thus in the neighborhood of $\{\omega\} \times X$ the bundle $M \to M/S^1$ used in Theorem 4.6 is:

$$\begin{aligned} M &= S\left(\mathbf{C}^n \times (P_{L^2} \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3))/S^1)\right) \to \\ &\to P\left(\mathbf{C}^n \times (P_{L^2} \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3))/S^1)\right) \end{aligned} \tag{6.6}$$

where the projection is the quotient by the S^1_{ST} action. In the above formula we have used the symbol "S" even though the fiber of the boundary of

$$\mathbf{C}^n \times (P_{L^2} \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3))/S^1)$$

is not a sphere, but $S^{2n-1} * SO(3)$, a space that is double covered by S^{2n+3}. For later computations we need to evaluate the powers of the first Chern class c_M of the fibration $M \to M/S^1_{\mathrm{ST}}$ on M/S^1.

When the $SO(3)$ fibration Fr_+X lifts to an $SU(2)$ fibration $\tilde{F}r_+X$ (i.e. when w_2 of the manifold X vanishes), then, following [17], we have an S^1 equivariant double cover map:

$$P_L \underset{X}{\times} (\tilde{F}r_+X \underset{SU(2)}{\times} cSU(2))/S^1 \to P_{L^2} \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3))/S^1$$

where S^1 acts in the standard way on $SU(2)$. The fibration

$$P_L \underset{X}{\times} (\tilde{F}r_+X \underset{SU(2)}{\times} cSU(2))/S^1$$

can be *orientation reversing* identified with the sphere bundle of the complex vector bundle $L^{-1} \otimes \Lambda^2_{+,\mathbf{C}}$ with the complex orientation. For the whole space M/S^1 we have the following:

Lemma 6.7 (Compare Proposition 2 in [17]) *If X is a four manifold with $w_2 = 0$, then there is a fiberwise covering map p of degree 2^n*

$$p : P\left(\mathbf{C}^n \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}})\right) \to P\left(\mathbf{C}^n \times (P_{L^2} \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3))/S^1)\right)$$

Moreover $p^*(c_M) = -2h_P$, where h_P is the positive generator of the fiber $\mathbf{C}P^{n+1}$ of

$$P\left(\mathbf{C}^n \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}})\right) \to X$$

Proof: In order to make the lift an S^1 equivariant we need to lift the S^1 action on P_{L^2} and on each fator of $\mathbf{C}^n = c(\underbrace{S^1 * \cdots * S^1}_{n \text{ times}})$. This gives us a 2^{n+1} cover of the fibrations before projectivization. Dividing by the S^1 action "cancels" one factor of 2.

On the fibers the map p is

$$\begin{tikzcd} S^{2n+3} \arrow[r] \arrow[d] & S^{2n-1} * SO(3) \arrow[d] \\ \mathbf{C}P^{n+1} \arrow[r, "p"] & (S^{2n-1} * SO(3)\,)/S^1 \end{tikzcd}$$

By using technics similar to the desingularization of the moduli space (4.7) and Theorem 4.14 we get:

$$\langle c_M^{n+1}, (S^{2n-1} * SO(3)\,)/S^1\rangle = \langle c_M^{n+1}, P(S^{2^C} \times SO(3)^C)\rangle = c_{SO(3)}[S^2] = 2$$

Hence

$$\langle p^*(c_M^{n+1}), \mathbf{C}P^{n+1}\rangle = 2^n\langle c_M^{n+1}, (S^{2n-1} * SO(3)\,)/S^1\rangle = 2^{n+1}\langle \gamma^{n+1}, \mathbf{C}P^{n+1}\rangle$$

giving the second conclusion modulo the sign. The minus in this formula is due to the fact that the identification of 2^n cover space with $P(\mathbf{C}^n \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}}))$ is orientation reversing.
□

Now we need to adjust the above results for the case of the space $N \cong \mathbf{C}P^n/\mathbf{Z}_p$.

6.2. SECOND STRATUM CONNECTIONS OVER THE ORBIFOLD N.

Let $\tau : \overline{\mathbf{C}P^2} \to N$ denote the quotient by $\mathbf{Z}_p$ action defined by $\tau[z_0 : z_1 : z_2] = [\exp(\frac{2\pi i}{p})z_0 : z_1 : z_2]$. This action has a fixed sphere $\{z_0 = 0\}$ and one fixed point $\{z_1 = z_2 = 0\}$. The quotient space N is not a manifold, having singularities at the fixed points of τ. Near the fixed point sphere N is diffeomorphic to $-p$ degree disc bundle over S^2, thus the singularity there can be resolved. The neighborhood of the fixed point is modelled on the open set in $\mathbf{R}^n$ divided by $\mathbf{Z}_p$ action with a single fixed point at the origin. Such spaces are called *orbifolds*. We will need the following properties of N:

Lemma 6.8 *Let $\sigma \in \mathrm{H}^2(N)$ be a cohomology class Poincare dual to the sphere with self–intersection $-p$. Also let $x \in \mathrm{H}^4(N)$ denote the generator of this group. Then we have*

$$\tau^*(\sigma) = p\gamma; \qquad \tau^*(x) = p\gamma^2$$

where $\gamma \in \mathrm{H}^2(\overline{\mathbf{C}P^2})$ is a generator of this group.

Proof: For the generator $x \in \mathrm{H}^4(N)$ we have $\tau^*(x) = p\gamma^2$, thus proving second equality. Also we have:

$$\langle \tau^*(\sigma^2), \overline{\mathbf{C}P^2} \rangle = p \cdot \langle \sigma^2, N \rangle = -p^2 = \langle (\pm p\gamma)^2, \overline{\mathbf{C}P^2} \rangle$$

Since $PD(\sigma)$ is a fixed by the action of τ, then we have the "+" sign in the first equality. $\square$

Over the orbifold N the line bundles L and the bundle $\Lambda^2_+(N)$ are defined as $\mathbf{Z}_p$ quotients of the corresponding bundles over $\overline{\mathbf{C}P^2}$. Thus if h_N denotes the class obtained by quotiening the generator h of the fiber $\mathbf{C}P^2$ of $P(\mathbf{C} \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}})) \to \overline{\mathbf{C}P^2}$, then we have:

$$(6.9) \quad \langle h_N^3 \pi^*(\sigma), P(\mathbf{C} \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}})) \rangle = \langle h_N \pi^*(\sigma), N \rangle = \frac{1}{p} \langle hp\gamma, \overline{\mathbf{C}P^2} \rangle = \langle h\gamma, \overline{\mathbf{C}P^2} \rangle$$

and

$$(6.10) \qquad \langle h_N^2 \pi^*(x), P(\mathbf{C} \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}})) \rangle = \langle \pi^*(x), N \rangle = \langle \gamma^2, \overline{\mathbf{C}P^2} \rangle$$

In particular this answer does not depend on the intersection number p. Now we are ready to come back to the computations of next examples.

Example 6.11 For the σ with the self–intersection -4, same as in the previous example, we want to increase the number of the classes σ appearing in the Donaldson invariant. According to Theorem 1.3 we expect the formula of the form:

$$(6.12) \qquad D(z\sigma^6) = a_1 D(z\sigma^2 x^2) + a_2 D(z\sigma^4 x) + a_3 D(zx) + a_4 D(z\sigma^2)$$

Each term in the above formula corresponds to different open set represented on the diagram:

$$\begin{tikzcd} 1 \arrow[d] \arrow[dr] & \\ 2 \arrow[dr] & 0 \arrow[d] \\ & 1 \end{tikzcd}$$

Solving the system of equations described in Theorem 5.2 gives us that the form

$$\phi^0 = \mu(\sigma)^6 - 5\mu(\sigma)^4 p(x_2) - (-4)\mu(\sigma)^2 p(x_1)^2$$

evaluates trivially on the sets $U^1_{1/4}$ and U^2_1. This shows that the coefficient $a_1 = -4 \cdot (-4)^2 = -64$, and $a_2 = 5 \cdot (-4) = -20$. Again it remains to evaluate ϕ^0 on U^0_1 and $U^1_{5/4}$. Similarly like in the previous example, in U^0_1 we have

$$\phi^0 = 4 \cdot \sigma^2[N] \cdot H[\Sigma SO(3)] \cdot p(x_1)[\mathcal{M}_k(Y, 0)] = 4 \cdot (-4) \cdot 2 \cdot (-4) D(zx) = 128 D(zx)$$

This gives us the coefficient at $D(zx)$ with the sign determined like in the previous example. Since the graph $\mathcal{J}$ terminates on one level below the set U_1^0, then we also get $\phi^1 = -32p(x_2)$. In $U^1_{5/4}$ we have:

$$\mu(\sigma) = \pi^*(\sigma) - h; \qquad\qquad p(x_i) = -4\pi^*(x_i) + h^2$$

Thus:

$$\begin{aligned}
\phi^0 &= (\pi^*(\sigma) - h)^6 - 5\,(\pi^*(\sigma) - h)^4 \cdot \left(-4\pi^*(x_i) + h^2\right) + 4\,(\pi^*(\sigma) - h)^2 \cdot \\
&\cdot \left(-4\pi^*(x_i) + h^2\right)^2 \\
&= h^6 - 6h^5\pi^*(\sigma) + 15h^4(\pi^*(\sigma))^2 - 5(h^4 - 4h^3\pi^*(\sigma) + 6h^2(\pi^*(\sigma))^2) \cdot \\
&\cdot (-4\pi^*(x_i) + h^2) + 4\left((\pi^*(\sigma))^2 - 2\pi^*(\sigma)h + h^2\right) \cdot (h^4 - 8h^2\pi^*(x_i)) \\
&= h^5\pi^*(\sigma)\,(-6 + 20 - 8) + h^4\pi^*(x_i)\left(15\sigma^2 + 20 - 30\sigma^2 + 4\sigma^2 - 32\right) \qquad (6.13)\\
&= 6h^5\pi^*(\sigma) + 32h^4\pi^*(x_i)
\end{aligned}$$

The set $U^1_{5/4}$ is $P(\mathcal{Q}_Y^C \times M^C)$, where M is the neighborhood of the connection $\omega \times N$ described in (6.6). Thus

$$\langle h^5\pi^*(\sigma), U^1_{5/4}\rangle = \langle c_Q, \mathcal{Q}_Y\rangle \cdot \langle h^3\pi^*(\sigma), M\rangle = D(\sigma^2) \cdot -\frac{1}{2} \cdot (-2)^3 \cdot \langle c_{Fr}\gamma, \overline{\mathbf{C}P^2}\rangle$$

Here $c_{Fr} = c_1(\bar{L} \otimes \Lambda^2_{+,C}) = -2\gamma$ and the extra minus in the front of the whole formula is due to the fact that the identification of the 2^n cover of M and $\bar{L} \otimes \Lambda^2_{+,C}$ was orientation reversing. Thus the contribution of the term $6h^5\pi^*(\sigma)$ in $U^1_{5/4}$ is $48D(\sigma^2)$. Similar computations give $\langle h^4\pi^*(x), U^1_{5/4}\rangle = -2D(\sigma^2)$. The contribution of ϕ^1 in $U^1_{1/4}$ is equal to $-32D(\sigma^2)$, thus adding these terms together we get:

$$D(z\sigma^6) = -64D(z\sigma^2x^2) - 20D(z\sigma^4x) - 128D(zx) - 48D(z\sigma^2)$$

Example 6.14 As the other way of generalizing the above considerations we shall prove that for $\sigma \cdot \sigma = -6$ we have

$$D(z\sigma^6) = -192D(zx) - 64D(z\sigma^2x^2) - 20D(z\sigma^4x) - 108D(z\sigma^2) + 720D_\sigma(z)$$

The graph $\mathcal{J}$ for this situation is

$$\begin{tikzcd}
1 \arrow[d] \arrow[dr] & \\
2 \arrow[d] \arrow[dr] & 0 \arrow[d] \\
3 & 1
\end{tikzcd}$$

and as before the form $\phi^0 = \mu(\sigma)^6 - 5\mu(\sigma)^4 p(x_2) - (-4)\mu(\sigma)^2 p(x_1)^2$ evaluates trivially on the sets $U^1_{1/6}$ and $U^2_{4/6}$, giving the coefficients at $D(z\sigma^2 x^2)$ and at $D(z\sigma^4 x)$. In $U^3_{3/2}$ we have:

$$\phi^0 = h[S^2] \cdot (3^6 - 5 \cdot 3^4 + 4 \cdot 3^3) = 2(729 - 405 + 36) = 720$$

which gives the constant at $D_\sigma(z)$ and $\phi^1 = -48p(x_2)$. Repeating the reasoning from the previous example we get that $\phi^0 = 4 \cdot (-6) \cdot 2 \cdot (-4) D(zx) = 192 D(zx)$ in U^0_1.

From (6.13) in $U^1_{7/6}$ we have:

$$\phi^0 = 6h^5 \pi^*(\sigma) + 54 h^4 \pi^*(x_i)$$

As the pull–back over $\overline{\mathbf{C}P^2}$ of the link of the reducible connection in $U^1_{7/6}$ does not depend on the self–intersection p of σ, we can repeat the computations from the previous example, obtaining:

$$\langle \phi^0, U^1_{7/6} \rangle + \langle \phi^1, U^1_{1/6} \rangle = (6 \cdot 8 + 54 \cdot (-2) - 48) \cdot D(\sigma^2) = -108 D(\sigma^2)$$

□

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