RIGID ANALYTIC GEOMETRY AND ABELIAN VARIETIES

MIHRAN PAPIKIAN

Abstract. The purpose of these notes is to introduce the basic notions of rigid analytic geometry, with the aim of discussing the non-archimedean uniformizations of certain abelian varieties.

1. Introduction

Complex analytic geometry is a powerful tool in the study of algebraic geometry over $\mathbb{C}$, especially with the help of Serre's GAGA theorems. For many arithmetic questions one would like to have a similar theory over other fields which are complete for a metric, for example, $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, or more generally, over non-archimedean fields. If one tries to use the metric of the field directly, then such a theory immediately runs into serious problems as the fields we are dealing with are totally disconnected.

It was John Tate who early 60's understood how to build the theory in such a way that one obtains reasonably well-behaved coherent sheaf theory and their cohomology [Tat71]. This theory of so-called rigid analytic spaces has many other results, which are strikingly similar with algebraic and complex analytic geometry. For example, one has the GAGA theorems over any non-archimedean ground field.

A very important development in the theory of rigid analytic spaces were the papers of David Mumford [Mum72b] and [Mum72a], where, using Grothendieck's idea of associating a "generic fibre" to a formal scheme, the author gives a theory of analytic uniformization of totally degenerate curves and totally degenerate abelian varieties. These results generalize Tate's work on non-archimedean uniformization of elliptic curves with split multiplicative reduction.

Over the last few decades rigid analytic geometry has developed into a fundamental tool in modern number theory. Among its impressive and diverse applications one could mention the following: Langlands correspondence relating automorphic and Galois representations [Dri74], [HT01], [LRS93], [Car90]; Ribet's proof that Shimura-Taniyama conjecture implies Fermat's Last Theorem [Rib90]; Degenerations of abelian varieties [CF90]; Fundamental groups of curves in positive characteristic [Ray94]; Construction of abelian varieties with a given endomorphism algebra [OvdP88]; and many others.

The purpose of these notes is to introduce some of the most basic notions of rigid analytic geometry, and, as an illustration of these ideas, to explain the analytic uniformization of abelian varieties with split purely toric reduction. The construction of such uniformizations is due to Tate and Mumford. In our exposition we follow [FvdP04] and especially §6 of *loc.cit.*

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2. Tate curve

As a motivation, and to have a concrete example in our later discussions, we start with elliptic curves. Here non-archimedean uniformization can be constructed very explicitly by using Weierstrass equation. Of course, such an equation-based approach does not generalize to higher dimensional abelian varieties, where one needs to have "well-behaved" p-adic spaces. This naturally leads to the notion of rigid analytic spaces. Later on we will return to the example of Tate curves, and see how Tate's theorem can be recovered from a more sophisticated point of view.

Let K be a field complete with respect to a non-archimedean valuation which we denote by $|\cdot|$. Let $R = \{x \in K \mid |x| \geq 1\}$ be the ring of integers in K, $\mathfrak{m} = \{x \in K \mid |x| > 1\}$ be the maximal ideal, and $k = R/\mathfrak{m}$ be the residue field. For example, we can take $K = \mathbb{Q}_p$ and $R = \mathbb{Z}_p$, or $K = \mathbb{F}_q(\!(t)\!)$ and $R = \mathbb{F}_q[\![t]\!]$. The field K need not necessarily be discretely valued, for example, it can be the completion of the algebraic closure of $\mathbb{Q}_p$.

Let E be an elliptic curve over $\mathbb{C}$. Then classically it is known that

$$E(\mathbb{C}) \cong \mathbb{C}/\Lambda \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau),$$

where $\mathrm{Im}(\tau) > 0$, with the isomorphism given in terms of the Weierstrass $\wp$-function and its derivative; see [Sil86, Ch.V]. To motivate the formulae over K, let

$$u = e^{2\pi i z}, \quad q = e^{2\pi i \tau} \quad \text{and} \quad q^{\mathbb{Z}} = \{q^k \mid k \in \mathbb{Z}\}.$$

Then there is a complex-analytic isomorphism

$$\begin{aligned} \mathbb{C}/\Lambda &\xrightarrow{\sim} \mathbb{C}^\times/q^{\mathbb{Z}} \\ z &\longmapsto u. \end{aligned}$$

Note that since $\mathrm{Im}(\tau) > 0$, we have $|q| < 1$. We use q-expansion to explicitly describe the isomorphism $E(\mathbb{C}) \cong \mathbb{C}^\times/q^{\mathbb{Z}}$.

Theorem 2.1. *Define quantities*

$$s_k(q) = \sum_{n\geq 1} \frac{n^k q^n}{1-q^n} \in \mathbb{Z}[\![q]\!],$$

$$a_4(q) = -5s_3(q), \quad a_6(q) = -\frac{5s_3(q) + 7s_5(q)}{12},$$

$$X(u,q) = \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2s_1(q),$$

$$Y(u,q) = \sum_{n\in\mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + s_1(q),$$

$$E_q : y^2 + xy = x^3 + a_4(q)x + a_6(q).$$

Then E_q is an elliptic curve, and X and Y define a complex analytic isomorphism

$$\begin{aligned} \mathbb{C}^\times/q^{\mathbb{Z}} &\longrightarrow E_q(\mathbb{C}) \\ u &\longmapsto \begin{cases} (X(u,q), Y(u,q)) & \text{if } u \notin q^{\mathbb{Z}}, \\ O & \text{if } u \in q^{\mathbb{Z}}. \end{cases} \end{aligned} \tag{2.1}$$

Proof. See [Sil94, Ch.V]. □

If we replace $\mathbb{C}$ by K and try to parametrize an elliptic curve E/K by a group of the form K/Λ, then we immediately run into a serious problem. Namely, K need not have non-trivial discrete subgroups. For example, if $\Lambda \subset \mathbb{Q}_p$ is any non-zero subgroup and $0 \neq t \in \Lambda$, then $p^n t \in \Lambda$ for all $n \geq 0$, so 0 is an accumulation point of Λ (in p-adic topology!). If the characteristic of K is positive then lattices do exist. For example, $\Lambda = \mathbb{F}_q[t]$ is an infinite discontinuous subgroup of $\mathbb{C}_\infty$, the completion of the algebraic closure of $\mathbb{F}_q(\!(\frac{1}{t})\!)$. However, the quotient $\mathbb{C}_\infty/\Lambda$ is very far from being an abelian variety. In fact one can show that $\mathbb{C}_\infty/\Lambda \cong \mathbb{C}_\infty$. Such quotients are very interesting since they naturally lead to the theory of Drinfeld modules [Dri74], but it is not what we want. Note in passing that unlike the situation over $\mathbb{C}$, we can take Λ to be a finitely generated discrete $\mathbb{F}_q[t]$-module of arbitrary rank in $\mathbb{C}_\infty$.

Tate's observation was that the situation can be salvaged, if one uses the multiplicative version of the uniformization. Indeed, $K^\times$ has lots of discrete subgroups, as any $q \in K^\times$ with $|q| < 1$ defines the discrete subgroup $q^\mathbb{Z} = \{q^n \mid n \in \mathbb{Z}\} \subset K^\times$.

Theorem 2.2 (Tate). *Let q be as above. Let $a_4(q), a_6(q)$ be defined as in Theorem 2.1. (This makes sense since $a_4(q), a_6(q) \in \mathbb{Z}[\![q]\!]$.) The series defining a_4 and a_6 converge in K. (Thanks to non-archimedean topology, $\sum_{n\geq 0} a_n$ is convergent if $|a_n| \to 0$.) Hence we can define E_q/K in terms of the Weierstrass equation as in Theorem 2.1. Then the map (2.1) defines a surjective homomorphism*

$$\phi : K^\times \to E_q(K)$$

with kernel $q^\mathbb{Z}$.

One reason why non-archimedean uniformizations are so useful for arithmetic applications (e.g. Langlands correspondence) is because they are compatible with the action of Galois. For example, in Theorem 2.2 we have $\phi(u^\sigma) = \phi(u)^\sigma$ for $\sigma \in \mathrm{Gal}(L/K)$ and L is a finite extension of K. Since σ maps the maximal ideal of the ring of integers of L to itself, it acts continuously on L and preserves the absolute value on L, i.e., $|\alpha^\sigma| = |\alpha|$. It follows from this that if $\sum \alpha_i$ is a convergent series with $\alpha_i \in L$, then $(\sum \alpha_i)^\sigma = \sum \alpha_i^\sigma$. Note that complex uniformization $\mathbb{C}^\times/q^\mathbb{Z} \to E(\mathbb{C})$ does not have this compatibility, as in general one cannot apply an element of Galois to the value of a convergent series by applying it to each term of the series (for example, write $\sqrt{2} = \sum \alpha_i$ with $\alpha_i \in \mathbb{Q}$ and take σ to be the generator of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$).

Over the complex numbers we know that every elliptic curve $E/\mathbb{C}$ is isomorphic to E_q for some $q \in \mathbb{C}^\times$. In the non-archimedean case, using q-expansions, we see that $a_4(q), a_6(q) \in \mathfrak{m}$ if $q \in \mathfrak{m}$. Hence $E_q \times k$ is given by $x^2 + xy = x^3$. In particular, E_q has split multiplicative reduction over R, so not every curve E over K can be isomorphic to E_q. A necessary condition is that E must have split multiplicative reduction over R. This is also a sufficient condition, i.e., if E has split multiplicative reduction then there is a unique $q \in K^\times$ with $|q| < 1$ such that $E_q \cong E$. For elliptic curves this last statement again can be proven by using the explicit nature of the uniformization: First, E/K has multiplicative reduction if and only if $|j(E)| > 1$. Next, one writes down the j-invariant of E_q as an infinite Laurent series in q with integer coefficients $j = \frac{1}{q} + 744 + 196884q + \cdots$, and, by carefully analyzing these series, shows that $j(q)$ defines a bijection between the two sets $\{q \in (K^{\mathrm{alg}})^\times \mid |q| < 1\}$ and $\{x \in K^{\mathrm{alg}} \mid |x| > 1\}$. Finally, one shows that E/K with $j(E) = \alpha$, $|\alpha| > 1$, is isomorphic over K to E_q with $j(q) = \alpha$ if and only if E

has split multiplicative reduction. To prove a similar statement for general abelian varieties one needs to use more sophisticated arguments. We will return to this in §4.5.

We would like to understand the higher dimensional generalizations of Theorem 2.2; we also would like to have an honest analytic object E^{an} with underlying set $E(K)$ and an isomorphism of analytic spaces $E^{\mathrm{an}} \cong \mathbb{G}_{m,K}^{\mathrm{an}}/q^{\mathbb{Z}}$ in an appropriate geometric category. To do all of this we need rigid analytic spaces – the subject of the next section.

3. Rigid analytic geometry

Suppose K is a field complete with respect to non-archimedean norm. We would like to do analytic geometry over K. The attempt of constructing such a theory in a straightforward manner, by using the topology on K, doesn't work since K is totally disconnected. To see why this is so, let $a \in K$ and let $D(a,r) = \{x \in K|\ |x-a| \leq r\}$ be the closed disc around a of radius r, where $r \in \mathbb{R}$ is a possible value of $|\cdot|$ on K. Let $D(a,r^-) = \{x \in K|\ |x-a| < r\}$ be the open disc, and $C(a,t) = \{x \in K|\ |x-a| = t\}$ be the circle of radius t around a. Then a counterintuitive thing happens, namely $C(a,t)$ turns out to be an open subset of K! Indeed, let $y \in C(a,t)$ then, due to the non-archimedean nature of the norm, $D(y,t^-) \subset C(a,t)$. Since $D(a,r) = \bigcup_{t\leq r} C(a,t)$, the disc $D(a,r)$ is both open and closed.

Tate's idea of constructing a theory of analytic spaces over K was to imitate the construction of algebraic varieties: An algebraic variety over a field K is obtained by gluing affine varieties over K with respect to Zariski topology. Further, the affine variety is the set of maximal ideals of some finitely generated algebra over K. Rigid (analytic) space over a complete non-archimedean valued field K are formed in a similar way. A rigid space is obtained by gluing affinoids with respect to a certain Grothendieck topology. One can think of an affinoid algebra over K as an algebra of functions defined on suitable subsets of K^d, in analogy with complex holomorphic functions defined on open subsets of $\mathbb{C}^d$. Alternatively, an affinoid algebra over K can be thought of as the completion of a finitely generated K-algebra. We note that the prime spectrum of an affinoid algebra is less relevant to the theory of rigid spaces because its usual Zariski topology cannot be used for glueing affinoid spaces.

We start by introducing the Tate algebra $T_n(K)$, which is the analogue of finitely generated polynomial algebra $K[z_1,\cdots,z_n]$. Then we define the notion of an affinoid, which, as we already mentioned, is the analogue of a finitely generated algebra over K. Finally, we define the notion of a rational subset, which is the analogue of basic open subvarieties $D(f) = \mathrm{Spec}(A_f)$. All this being done, we "glue" the affinoids according to a recipe of Grothendieck to construct rigid analytic spaces. The standard references for this section are [BGR84] and [FvdP04].

3.1. **Tate algebra.** The *Tate algebra* $T_n(K)$ is the algebra of formal power series in $X_1, X_2, \ldots, X_n$ with coefficients in K satisfying the following condition

$$T_n(K) = K\langle X_1,\ldots,X_n\rangle = \left\{ \sum_{k\in\mathbb{N}^n} a_k X_1^{k_1}\cdots X_n^{k_n} \;\middle|\; \lim_{k_1+\cdots+k_n\to\infty} |a_k| = 0 \right\}.$$

This algebra is provided with the norm:

$$\left\| \sum_{k \in \mathbb{N}^n} a_k X_1^{k_1} \cdots X_n^{k_n} \right\| = \max \left\{ |a_k| \;\middle|\; k \in \mathbb{N}^n \right\}.$$

Proposition 3.1. *$T_n(K)$ is a Banach algebra with respect to this norm. That is, for any $a, b \in T_n$ and any $\alpha \in K$ we have*

1. $\|a\| \geq 0$ *with equality if and only if* $a = 0$,
2. $\|a + b\| \leq \max(\|a\|, \|b\|)$,
3. $\|\alpha \cdot a\| = |\alpha| \cdot \|a\|$,
4. $\|1\| = 1$,
5. $\|a \cdot b\| \leq \|a\| \cdot \|b\|$,
6. T_n *is complete with respect to* $\|\cdot\|$.

Recall that if B is a Banach algebra over $\mathbb{C}$ then the set of maximal ideals of A, $\mathrm{Max}(A)$, has a natural structure of compact Hausdorff space. Moreover, by Gelfand-Mazur theorem, for each $x \in \mathrm{Max}(A)$ we have $A/x \cong \mathbb{C}$.

Some of the statements in the next proposition can be proven by using the analogue of Weierstrass preparation theorem.

Proposition 3.2. (a) *Every ideal in T_n is closed.* (b) *T_n is Noetherian domain.* (c) *If $I \subset T_n$ is an ideal then there exists a finite injective morphism $T_d \to T_n/I$ with d the Krull dimension of T_n/I.* (d) *If $I \subset T_n$ is a maximal ideal then T_n/I is a finite extension of K and the valuation on K uniquely extends to T_n/I.* (e) *T_n is a Jacobson ring, i.e., the radical of an ideal I is equal to the intersection of all the maximal ideals containing I.*

For $f \in T_n$ and $x \in \mathrm{Max}(T_n(K))$ let $f(x)$ be the image of f in the field T_n/x. Since by the previous proposition we can (uniquely) extend the norm on K to T_n/x, we can define the *spectral norm*

$$\|f\|_{\mathrm{sp}} = \sup\{|f(x)| \mid x \in \mathrm{Max}(T_n)\}.$$

Theorem 3.3. *For any $f \in T_n(K)$ we have $\|f\| = \|f\|_{\mathrm{sp}}$.*

This is a consequence of the maximum modulus principle, which says that there is $x \in \mathrm{Max}(T_n)$ such that $\|f\|_{\mathrm{sp}} = |f(x)|$.

3.2. **Affinoid algebras.** Let $I \subset T_n$ be an ideal. The algebra $A = T_n/I$ is called an *affinoid algebra*. Since I is closed in T_n, the algebra A is noetherian and is a Banach algebra with respect to the quotient norm on A, i.e., $\|\bar{f}\| = \inf\{\|f + g\| \mid g \in I\}$. For simplicity of the exposition we will always assume that A is reduced.

Let $X = Z(I) \subset \mathrm{Max}(T_n)$ be the zero set of I. By restriction, the elements of T_n can be regarded as a function on Y. Since T_n is Jacobson and A is assumed to be reduced, the element $f \in T_n$ is 0 on Y if and only if $f \in I$. In particular, the elements of A can then be regarded as (holomorphic) functions on X.

The canonical surjection $T_n \to A$ induces a map between the sets of maximal ideals $\mathrm{Max}(A) \to \mathrm{Max}(T_n)$. This map identifies X and $\mathrm{Max}(A)$. It follows that the elements of A can be regarded as functions on $\mathrm{Max}(A)$. The set $X = \mathrm{Max}(A)$ is provided with the topology generated by the subsets $U \subset X$ such that there is some homomorphism of affinoid algebras $\varphi : A \to B$ with $U = \varphi^*(\mathrm{Max}(B))$. (It can be shown that all maps of K-affinoids are automatically continuous.) The subsets of X of this type are called *open affinoid subsets*.

Definition 3.4. Let $f_0, f_1, \dots, f_n \in A$ be elements such that $A = f_0A + \cdots + f_nA$. This means that $f_0, f_1, \dots, f_n$ have no common zeros on $X = \mathrm{Max}(A)$. The set

$$X(f_0, \dots, f_n) = \{x \in X \mid |f_i(x)| \leq |f_0(x)|\}$$

is called a *rational subset* of X.

Theorem 3.5. *$X(f_0, \dots, f_n)$ is an open affinoid subset of X. The corresponding affinoid algebra is*

$$A(f_0, \dots, f_n) = A\langle t_1, \dots, t_n\rangle/(f_0t_i - f_i \mid i = 1, \dots, n).$$

If R_1 and R_2 are rational subsets of X then $R_1 \cap R_2$ is also a rational subset of X. If $R_1 \subset X$ is rational and $R_2 \subset R_1$ is rational then R_2 is a rational subset of X. Every open affinoid subset of X is a finite union of rational subsets of X.

In order to get a structure of a ringed space on X we need the notion of Grothendieck's topology.

Definition 3.6. A *Grothendieck topology* on a topological space X consists of following data:

(1) A family $\mathcal{G}$ of open subsets of X satisfying the following conditions:
   (a) $\emptyset, X \in \mathcal{G}$;
   (b) For any U and V in $\mathcal{G}$ the intersection $U \cap V$ is also in $\mathcal{G}$.
(2) For all $U \in \mathcal{G}$ a set $\mathrm{Cov}(U)$ of coverings of U such that
   (a) For all $\mathcal{U} \in \mathrm{Cov}(U)$ and all $V \in \mathcal{U}$, $V \in \mathcal{G}$;
   (b) $\{U\} \in \mathrm{Cov}(U)$;
   (c) If $\mathcal{U} \in \mathrm{Cov}(U)$ and if $V \subset U$ with $V \in \mathcal{G}$ then

$$\mathcal{U} \cap V = \{U' \cap V \mid U' \in \mathcal{U}\} \in \mathrm{Cov}(V);$$

   (d) If $(U_i)_{i \in I} \in \mathrm{Cov}(U)$ and if $\mathcal{U}_i \in \mathrm{Cov}(U_i)$ then

$$\bigcup_{i \in I} \mathcal{U}_i = \{U' \mid \text{there is } i \in I \text{with } U' \in \mathcal{U}_i\} \in \mathrm{Cov}(U).$$

The elements of $\mathcal{G}$ are called *admissible subsets* of X. Elements in $\mathrm{Cov}(U)$ are called *admissible coverings* of U.

Given a topological space endowed with Grothendieck topology, one defines in the usual manner, using the admissible opens and admissible coverings, (pre)sheaves on this space and Cech cohomology for a presheaf.

Let A be an affinoid algebra. Then by Theorem 3.5 we can endow $X = \mathrm{Max}(A)$ with a Grothendieck topology, where the admissible opens are the rational subsets and admissible coverings are the *finite* coverings by rational subsets. Denote by $\mathcal{O}(U)$ the affinoid algebra of a rational subset $U \subset X$. The K-algebra $\mathcal{O}(U)$, together with canonical restriction maps define the presheaf $\mathcal{O}$ on X. Tate proved that $\mathcal{O}$ is in fact a sheaf (he proved that for any $\mathcal{U} \in \mathrm{Cov}(X)$ the Cech cohomology groups $\mathrm{H}^n(\mathcal{U}, \mathcal{O}) = 0$ for $n > 0$, and $\mathrm{H}^0(\mathcal{U}, \mathcal{O}) = A$).

Definition 3.7. An *affinoid space* $\mathrm{Sp}(A)$ is a topological space $X = \mathrm{Max}(A)$ with A an affinoid algebra, provided with Grothendieck topology of rational subsets and the structure sheaf $\mathcal{O}$.

Let $X = \mathrm{Sp}(A)$. Let $\{U_\alpha = \mathrm{Sp}(A_\alpha)\}$ be a collection of affinoid subdomains, and let $U = \bigcup_\alpha U_\alpha$. We say that U is an *admissible open* (and $\{U_\alpha\}$ form an *admissible affinoid covering* of U) if for any morphism $\varphi : Y = \mathrm{Sp}(B) \to X$ with $\varphi(Y) \subseteq U$, the collection $\{\varphi^{-1}(U_\alpha)\}$ of affinoid subdomains of Y has a finite subcollection covering Y.

Definition 3.8. A *rigid analytic space* is a triple $(X, \mathcal{G}, \mathcal{O})$ where X is a topological space, provided with a Grothendieck topology $\mathcal{G}$ and a sheaf of K-algebras $\mathcal{O}$, such that there exists a covering $(X_i)_{i\in I} \in \mathrm{Cov}(X)$ with $(X_i, \mathcal{G}|_{X_i}, \mathcal{O}|_{X_i})$ being an affinoid space.

Example 3.9. Let $\mathbb{B}^n = \mathrm{Sp}\ K\langle T_1, \dots, T_n\rangle$. Then

$$\mathbb{B}^n(K^{\mathrm{alg}}) = \{(t_1, \dots, t_n) \in (K^{\mathrm{alg}})^n \mid |t_i| \le 1\}/\mathrm{Aut}(K^{\mathrm{alg}}/K).$$

Indeed, any $(t_1, \dots, t_n)$ can be obtained from some homomorphism

$$K\langle T_1, \dots, T_n\rangle \to K' \subset K^{\mathrm{alg}}$$

and vice-versa. Hence $\mathbb{B}^n$ is the rigid-analytic n-dimensional unit ball.

Let $r_1 < r_2 < \cdots < 1$ be real numbers in the image of the norm on K^{alg} such that r_i converge to 1. Let $U = \{\mathbf{t} \in \mathbb{B}^n \mid |t_i| < 1 \text{ for all } i\}$. Let $U_j = \{\mathbf{t} \in \mathbb{B}^n \mid |t_i| \le r_j \text{ for all } i\}$. It is easy to check that $\{U_j\}$ form an admissible affinoid covering of U; hence U is an admissible open. On the other hand, let $\partial\mathbb{B}^n = \{\mathbf{t} \mid |t_i| = 1\}$. Then

$$\begin{aligned}\partial\mathbb{B}^n &= \mathrm{Sp}\ K\langle T_1, T_1^{-1} \dots, T_n, T_n^{-1}\rangle \\ &= \mathrm{Sp}\ K\langle T_1, \dots, T_n, U_1, \dots, U_n\rangle/(T_1U_1 - 1, \dots, T_nU_n - 1)\end{aligned}$$

is an affinoid subdomain. This latter affinoid ring can also be described as the ring

$$\left\{ f = \sum_{i\in\mathbb{Z}^n} a_i \mathbf{t}^i \mid |a_i| \to 0 \text{ as } |\sum i| \to \infty \right\}.$$

Consider the covering $\mathbb{B} = U \cup \partial\mathbb{B}$. If we allow such coverings, then $\mathbb{B}$ will be disconnected (something we do not want). Fortunately, according to our definition of admissible coverings, $\mathbb{B} = U \cup \partial\mathbb{B}$ is not an admissible covering of $\mathbb{B}$. Indeed, consider the identity map from the affinoid $\mathbb{B}$ to itself. Then $\mathbb{B} = (\cup_j U_j) \bigcup \partial\mathbb{B}$ has no finite subcovering. So the notion of admissible opens and admissible coverings is introduced to rule out such unplesant phenomena as the unit ball being disconnected.

Example 3.10. Let $X = \mathrm{Spec}\ K[Z_1, \dots, Z_n]/(f_1, \dots, f_s)$ be a separated reduced affine scheme of finite type over K. Assume K is algebraically closed. The topological space X can be identified with a closed (in Zariski topology) subset of K^n:

$$X = \{\mathbf{z} = (z_1, \dots, z_n) \in K^n \mid 0 = f_1(\mathbf{z}) = \cdots = f_s(\mathbf{z})\}.$$

Choose $\pi \in K^\times$ with $|\pi| < 1$. Let for $m \ge 1$

$$X_m = \{\mathbf{z} \in X \mid |z_i| \le |\pi|^{-m} \text{for } 1 \le i \le n.\}$$

X_m has a natural structure of an affinoid:

$$X_m = \mathrm{Sp}\ K\langle \pi^m Z_1, \dots, \pi^m Z_n\rangle/(f_1, \dots, f_s).$$

The affinoids X_m can be glued together in an obvious manner, where X_m is rational in X_{m+1}. This gives a rigid analytic space X^{an} for which the structure sheaf $\mathcal{O}(X^{\mathrm{an}})$ is $\varprojlim \mathcal{O}(X_m)$. One can check that X^{an} is well-defined and intrinsic for X.

As a more concrete example, let's consider $(\mathbb{A}^1_K)^{\mathrm{an}}$. Pick $c \in K^\times$ with $|c| > 1$. Let $\mathbb{B}_r = \mathrm{Sp}\ K\langle \frac{t}{c^r}\rangle = \{|t| \leq |c|^r\}$ for $r = 0, 1, 2, \dots$ Then

$$\begin{gathered}\mathbb{B}_0 \hookrightarrow \mathbb{B}_1 \hookrightarrow \cdots \mathbb{B}_r \hookrightarrow \mathbb{B}_{r+1} \hookrightarrow \cdots \\ t_r/c \leftarrow t_{r+1}\end{gathered}$$

are glued together as indicated to give $\mathbb{A}^{1,\mathrm{an}}_K$.

Now given a general algebraic variety X over K we can associate a rigid analytic space X^{an} by considering an open affine cover of X, and using the fact that the intersection of two open affines in X is again an open affine. So we can analytify each open affine and then glue them along the intersections.

Of course, one can show that $X \to X^{\mathrm{an}}$ is a functor from the category of separated reduced schemes of finite type over a field into the category of rigid analytic spaces. It is fully faithful on proper schemes.

Example 3.11. Now let's return to Tate curve example and consider it from rigid-analytic point of view. Let $\mathbb{G}_{m,K} = \mathrm{Spec}\ K[X,Y]/(XY-1)$ be the algebraic torus, and let $\mathbb{G}^{\mathrm{an}}_{m,K}$ be its analytification. Let $q \in K^\times$ with $|q| < 1$. We can identify q with the automorphism of the rigid space $\mathbb{G}^{\mathrm{an}}_{m,K}$, given by $z \mapsto zq$. The action of the group $\Gamma = q^{\mathbb{Z}}$ of the rigid space $\mathbb{G}^{\mathrm{an}}_{m,K}$ is *discontinuous*. This means that $\mathbb{G}^{\mathrm{an}}_{m,K}$ has an admissible covering $\{X_i\}$ such that for each i, the set $\{\gamma \in \Gamma \mid \gamma X_i \cap X_i \neq \emptyset\}$ is finite. An example of such a covering is $\{X_i\}_{i\in\mathbb{Z}}$ where the affinoid subset X_n of $\mathbb{G}^{\mathrm{an}}_{m,K}$ is given by the inequalities $X_n = \{x \in \mathbb{G}^{\mathrm{an}}_{m,K} \mid |q|^{(n+1)/2} \leq |z(x)| \leq |q|^{n/2}\}$. Note that $X_n \cap X_{n-1} = \{x \in \mathbb{G}^{\mathrm{an}}_{m,K} \mid |z(x)| = |q|^{n/2}\} = \mathrm{Sp}\ K\left\langle \frac{t}{|q|^{n/2}}, \frac{|q|^{n/2}}{t}\right\rangle$. Now we can form the quotient $\mathcal{E} = \mathbb{G}^{\mathrm{an}}_{m,K}/\Gamma$ in a usual manner, i.e., by gluing annuli appropriately. The Grothendieck topology on $\mathcal{E}$ is defined as follows: $U \subset \mathcal{E}$ is admissible if there is an affinoid $V \subset \mathbb{G}^{\mathrm{an}}_{m,K}$ such that under the canonical projection $\mathbb{G}^{\mathrm{an}}_{m,K} \to \mathcal{E}$ the affinoid V maps bijectively to U. For example, X_0 and X_1 map bijectively into $\mathcal{E}$ and form a covering of this latter space. The intersection $U_0 \cap U_1$ is the image of $(X_{-1} \cap X_0) \bigcup (X_0 \cap X_1)$. The structure sheaf $\mathcal{O}$ on $\mathcal{E}$ is defined by $\mathcal{O}(\emptyset) = \{0\}$, $\mathcal{O}(\mathcal{E}) = K$ and $\mathcal{O}(U_i) = \mathcal{O}(X_i)$, $\mathcal{O}(U_0 \cap U_1) = \mathcal{O}(X_{-1} \cap X_0) \oplus \mathcal{O}(X_0 \cap X_1)$. One checks that this gives a well-defined structure of a rigid analytic space on $\mathcal{E}$.

To recover Tate's theorem 2.2 one has to study the field of meromorphic functions $\mathcal{M}(\mathcal{E})$ on $\mathcal{E}$. After proving a Riemann-Roch theorem for $\mathcal{M}(\mathcal{E})$, it is not hard to show that $\mathcal{M}(\mathcal{E})$ is the field of rational functions of an algebraic elliptic curve E. In particular, $\mathcal{E} = E^{\mathrm{an}}$. To get the Weierstrass equation of E one observes that by construction the field $\mathcal{M}(\mathcal{E})$ is isomorphic to the subfield of $\mathcal{M}(\mathbb{G}^{\mathrm{an}}_{m,K})$ fixed by Γ, i.e., the functions which are invariant under the substitution $f(z) \mapsto f(qz)$. In particular, the functions $X(u,q)$ and $Y(u,q)$ in Theorem 2.1 have this property. Analyzing the poles and zeros of these functions, one gets from the Riemann-Roch $\mathcal{M}(\mathcal{E}) = K(X,Y)$ and E is given by the equation in Theorem 2.1. For the details see [FvdP04, §5.1].

4. Uniformization of abelian varieties

In this section we study the quotients of $(\mathbb{G}^{\mathrm{an}}_{m,K})^g$ by discrete rank-g lattices, and the questions of algebraicity of such quotients. As a motivation we first recall the well-known corresponding theorem over $\mathbb{C}$.

4.1. **The complex case.** Let $V \cong \mathbb{C}^g$ be a finite-dimensional complex vector space. A map $H : V \times V \to \mathbb{C}$ is a *Hermitian form* if it is linear in the first variable, anti-linear in the second variable, and $H(u,v) = \overline{H(v,u)}$. The Hermitian form H is *positive-definite* if $H(u,u) > 0$ for all $u \in V \setminus \{0\}$. By a *lattice* Λ in V we mean a discrete subgroup isomorphic to $\mathbb{Z}^{2g}$. A *Riemann form* on $G = V/\Lambda$ is a positive-definite Hermitian form on V such that $\mathrm{Im}(H)$ is $\mathbb{Z}$-valued on $\Lambda \times \Lambda$.

Theorem 4.1. *The quotient $G = V/\Lambda$ is the analytification of a complex abelian variety if and only if G possesses a Riemann form.*

The existence of a Riemann form is a rather strong restriction on G when $\dim_{\mathbb{C}} V \geq 2$, hence not all G are algebraic. The idea behind this classical result is that G can only be an abelian variety if there exists an analytic line bundle $\mathcal{L}$ on G, which provides an embedding of G into some projective space $\mathbb{P}^N_{\mathbb{C}}$. Then the image of G is an algebraic variety according to Serre's GAGA. Since obviously it is also smooth projective group variety, G is an abelian variety. The existence of $\mathcal{L}$ is equivalent to the existence of a line bundle $\widetilde{\mathcal{L}}$ on $V = \mathbb{C}^g$ which carries an action of Λ and such that $\widetilde{\mathcal{L}}^{\otimes 3}$ has enough Λ-invariant sections. This then translates into the existence of a Riemann form. For the details we refer to [Mum70, Ch.1].

4.2. **Non-archimedean case.** As with the Tate curves, we consider multiplicative version of the construction in §4.1.

Let $T = \mathbb{G}^g_{m,K} = \mathrm{Spec}\ K[x_1, x_1^{-1}, \ldots, x_g, x_g^{-1}]$ be the algebraic torus. Let T^{an} be the rigid analytic space corresponding to the analytification of T. There is a natural group homomorphism $\ell : T^{\mathrm{an}} \to \mathbb{R}$ given by $\ell(x) = (-\log|z_1|, \ldots, -\log|z_g|)$. A lattice Λ is a torsion-free subgroup of $T(K)$ (of maximal $\mathbb{Z}$-rank g) which is discrete in T^{an}. That is, the intersection of each affinoid with Λ is finite. Equivalently, $\ell(\Lambda)$ is a rank-g lattice in $\mathbb{R}^g$. We would like to give $G = T^{\mathrm{an}}/\Lambda$ a structure of a rigid analytic space.

For simplicity assume the valuation on K is discrete, and choose a basis for T^{an} and $\mathbb{R}^g$ such that $\ell(\Lambda) = \mathbb{Z}^g$. Consider the standard cube $S := \{(x_1, \ldots, x_g) \in \mathbb{R}^g \mid |x_i| \leq 1/2 \text{ for all } i\}$. Then $\mathbb{R}^g$ is covered by $\{a + S\}$ with $a \in \mathbb{Z}^g$. Now the set $U := \ell^{-1}(S)$ is an affinoid subspace of T^{an}. The translates by elements of Λ, $\lambda U := \ell^{-1}(\ell(\lambda) + S)$, are again affinoid subsets, and give an admissible covering $T^{\mathrm{an}} = \bigcup_{\lambda \in \Lambda} \lambda U$. We use the covering of G by the images $V_\lambda := \mathrm{pr}(\lambda U)$ to endow it with a structure of a rigid analytic space. There is a notion of *proper* rigid analytic space, and it is not hard to show that the group space G is proper. For the sake of completeness we recall the definition.

Definition 4.2. The rigid space X is said to be *separated* if there is an affinoid covering $\{Y_i\}$ such that every $Y_i \cap Y_j$ is again an affinoid and $\mathcal{O}(Y_i) \widehat{\otimes}_K \mathcal{O}(Y_j) \to \mathcal{O}(Y_i \cap Y_j)$ is surjective. Here the continuous tensor $\widehat{\otimes}$ on two K-affinoid algebras $A = K\langle x_1, \ldots, x_n\rangle$ and $B = K\langle y_1, \ldots, y_m\rangle$ is defined by

$$A \widehat{\otimes}_K B = K\langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle.$$

An affinoid subset Y_1 of an affinoid space Y_2 is said to lie in the *interior* of Y_2, and this property is denoted by $Y_1 \subset\subset Y_2$, if $Y_2 = \mathrm{Sp}(A)$ and A has a presentation of the form $A = K\langle z_1, \ldots, z_n\rangle = K\langle Z_1, \ldots, Z_n\rangle/(f_1, \ldots, f_s)$ and there exists a suitable $\rho < 1$ such that $Y_1 \subset \{y \in Y_2 \mid \text{ for all } |z_i(y)| \leq \rho\}$. A separated rigid space is *proper*, if there exist two finite admissible affinoid coverings $\{X_i\}_{i=1,\ldots,n}$ and $\{X'_i\}_{i=1,\ldots,n}$ with $X_i \subset\subset X'_i$ for all i.

To see that G is proper, take two coverings of T^{an} which come from the process described above by taking in $\mathbb{R}^g$ two cubes of side length 1 and 2. Then we have $\lambda U^{(1)} \subset\subset \lambda U^{(2)}$.

The Riemann form condition in the rigid setting translates into the following:

Theorem 4.3. *G is an abelian variety if and only if there is a homomorphism*

$$H : \Lambda \to \Lambda^\vee = \mathrm{Hom}(T^{\mathrm{an}}, \mathbb{G}_{m,K}^{\mathrm{an}})$$

such that $H(\lambda)(\lambda') = H(\lambda')(\lambda)$, and the symmetric bilinear form

$$\langle \lambda, \lambda' \rangle := -\log |H(\lambda')(\lambda)|$$

on $\Lambda \times \Lambda$ is positive definite.

The strategy of the proof of this theorem is very similar to the one over $\mathbb{C}$. First, we have the following analogue of the *Appell-Humbert theorem* over $\mathbb{C}$.

Proposition 4.4. *There is a functorial isomorphism of groups*

$$\mathrm{Pic}(G) \cong \mathrm{H}^1(\Lambda, \mathcal{O}(T^{\mathrm{an}})^\times),$$

where $\mathcal{O}(T^{\mathrm{an}})^\times = \{\beta \cdot z_1^{\alpha_1} \cdots z_g^{\alpha_g} \mid \beta \in K^\times \text{ and } \underline{\alpha} \in \mathbb{Z}^g\}$ is the multiplicative group of nowhere vanishing holomorphic functions on T^{an}, and $\Lambda \subset T^{\mathrm{an}}(K)$ acts through its translation action on T^{an}. Moreover, every element in $\mathrm{H}^1(\Lambda, \mathcal{O}(T^{\mathrm{an}})^\times)$ can be uniquely represented by $Z_\lambda(z) = d(\lambda)H(\lambda)(z)$, where

$$H : \Lambda \to \Lambda^\vee = \{z_1^{\alpha_1} \cdots z_g^{\alpha_g} \mid \underline{\alpha} \in \mathbb{Z}^g\}$$

is a group homomorphism and $d : \Lambda \to K^\times$ is a morphism satisfying

$$d(\lambda_1\lambda_2)d(\lambda_1)^{-1}d(\lambda_2)^{-1} = H(\lambda_2)(\lambda_1).$$

Proof. The proof is essentially the same as over $\mathbb{C}$ [Mum70, Ch.1], using the fact [FvdP04, Ch.VI] that the line bundles on T^{an} are trivial. $\square$

So every line bundle L on G corresponds to a cocycle Z_λ, and every such cocycle is given by a pair (H, d). So every line bundle corresponds to a pair (H, d), and we will denote this line bundle by $L(H, d)$. One easily checks that $L(H_1, d_1) \cong L(H_2, d_2)$ if and only if $H_1 = H_2$ and $d_1(\lambda) = \lambda^{\underline{\alpha}} d_2(\lambda)$ for some $\underline{\alpha} \in \mathbb{Z}^g$.

To have an embedding of G into some $\mathbb{P}_K^{n,\mathrm{an}}$, one needs an ample line bundle L. The global section of a given line bundle can be explicitly described using theta series and their Fourier expansions (this relies on the explicit description of L given above). It turns out that $L(H, d)$ is ample if and only if $-\log |H|$ is positive definite. The positive definiteness is used to show that enough explicit formal Fourier series are convergent, and hence give honest sections, so that $L^{\otimes 3}$ is very ample. Finally, one uses rigid-analytic GAGA theorems, proved by Kiehl, to conclude that G is algebraic (that is, it is known that every closed analytic subspace of $\mathbb{P}_K^{n,\mathrm{an}}$ is the analytification of a closed subspace of $\mathbb{P}_K^n$).

It is natural to ask whether, given an abelian variety A of dimension g over K, there is an isomorphism $A^{\mathrm{an}} \cong T^{\mathrm{an}}/\Lambda$ for some lattice Λ. The corresponding question over $\mathbb{C}$ is known to have an affirmative answer. Indeed, let $T_A = \mathrm{H}^0(A, \Omega^1)^\vee$ be the tangent space to A at the identity, and let $\Lambda_A = \mathrm{H}_1(A, \mathbb{Z})$. There is a natural injective homomorphism

$$\Lambda_A \to T_A$$
$$\gamma \mapsto (\omega \mapsto \int_\gamma \omega),$$

where $\omega \in \mathrm{H}^0(A, \Omega^1)$. The quotient T_A/Λ_A can be identified with $A(\mathbb{C})$; see [Mum70, Ch.1].

We already saw on the example of the Tate curve that the corresponding question over non-archimedean fields has a more complicated answer. To characterize exactly the abelian varieties which are quotients of analytic tori one needs three fundamental concepts: Néron models, formal schemes with their "generic fibre" functor, and the notion of analytic reduction of a rigid analytic space. We will only outline the ideas involved, with the details being somewhat technical.

4.3. **Analytic reductions.** We assume that K has discrete valuation, R is its ring of integers, $\mathfrak{m}$ the maximal ideal in R, and k is the residue field.

Given a rigid analytic space X and an admissible affinoid covering $\mathcal{U}$, one can associate to it a locally finite type scheme $\overline{X}$ over k, called the *analytic reduction* of X with respect to $\mathcal{U}$.

First, let $X = \mathrm{Sp}(A)$ and assume A is reduced. Denote by A° the R-algebra $\{a \in A \mid \|a\| \le 1\}$. There is a natural ideal $A^{\circ\circ} = \{a \in A \mid \|a\| < 1\}$. The canonical analytic reduction of X is the k-scheme $\overline{X} = \mathrm{Spec}(A^\circ/A^{\circ\circ})$.

Next, if X is an analytic space and if $\mathcal{U} = (U_i)_{i\in I} \in \mathrm{Cov}(X)$ has certain good properties then the canonical reductions $\overline{U}_i$ can be glued together to a k-scheme $\overline{X}$ with a surjective map $X \to \overline{X}$. Instead of going into technical details clarifying the relation between the properties of X and $\overline{X}$, we give few relevant examples.

Example 4.5. Let $A = T_n = K\langle Z_1, \dots, Z_n\rangle$ be the Tate algebra. Then $A^\circ = R\langle Z_1, \dots, Z_n\rangle$, and it is easy to see that $\overline{A} = k[Z_1, \dots, Z_n]$.

Example 4.6. Let $X = \{x \in K \mid |q| \le |x| \le 1\}$ with $q \in K^\times$ and $|q| < 1$. We have $A = K\langle Y, Z\rangle/(YZ - q) = K\langle Z, \frac{q}{Z}\rangle$. Then $\overline{A} = k[w, u]/wu$, where w is the image of Z and u is the image of $\frac{q}{Z}$. So the analytic reduction $\overline{X}$ consists of two affine lines ℓ_1 and ℓ_2 intersecting (transversally) at one point P. The points in X with $|x| = 1$ are mapped onto $\ell_1 - P$ and the points which satisfy $|x| = |q|$ are mapped onto $\ell_2 - P$. Finally, the points which satisfy $|q| < |x| < 1$ are mapped to P.

Example 4.7. Let $X = \mathbb{G}_{m,K}^{\mathrm{an}}$. Take $q \in K^\times$ with $|q| < 1$. Take the affinoid covering $\bigcup_{n\in\mathbb{Z}} X_n$, where $X_n = \{x \in K \mid |q|^n \le |x| \le |q|^{n-1}\}$. Then $\overline{X}$ is an infinite chain of $\mathbb{P}^1_k$. This follows from the previous example. Indeed, $\overline{X}_n$ consists of two affine lines $\ell_{1,n}$ and $\ell_{2,n}$ intersecting at a point P_n. The reduction $\overline{X}_{n+1}$ glued to $\overline{X}_n$ by identifying $\ell_{2,n} - P_n$ and $\ell_{1,n+1} - P_{n+1}$ so that there results $\mathbb{P}^1_k$ with $\mathbb{P}^1_k - 0 = \ell_{2,n} - P_n$ and $\mathbb{P}^1_k - \infty = \ell_{1,n+1} - P_{n+1}$.

Example 4.8. The analytic reduction of $T^{\mathrm{an}} = (\mathbb{G}_{m,K}^{\mathrm{an}})^g$ is the product situation of the previous example. Namely, there is an admissible covering of T^{an} such that $\overline{T^{\mathrm{an}}} = \overline{X}^n$ with $X = \mathbb{G}_{m,K}^{\mathrm{an}}$. Each irreducible component of $\overline{T^{\mathrm{an}}}$ is isomorphic to the cross-product of g copies of $\mathbb{P}^1_k$. The irreducible components are glued together along unions of coordinate hyperplanes in k^g. In particular, every singular point in $\overline{T^{\mathrm{an}}}$ is locally isomorphic to a singularity in the intersection of d coordinate hyperplanes in k^g with $d \leq g$. If we remove the singular locus from $\overline{T^{\mathrm{an}}}$ then we get $\bigcup_{\alpha\in\mathbb{Z}^g}(\mathbb{G}_{m,k}^g)_\alpha$, i.e., a disjoint union of copies of $\mathbb{G}_{m,k}^g$ labelled by g-tuples of integers.

Example 4.9. Let $G = T^{\mathrm{an}}/\Lambda$. By construction, the affinoid covering $\mathcal{U}$ used in the previous example is preserved under the action of Λ. Hence the group Λ acts equivariantly on the analytic reduction $T^{\mathrm{an}} \to \overline{T^{\mathrm{an}}}$. Hence the analytic reduction of T^{an}/Λ is a finite union of copies of g-tuple products $(\mathbb{P}_k^1)\times\cdots\times(\mathbb{P}_k^1)$ glued together along coordinate hyperplanes, and the smooth locus of $\overline{G}$ is an extension of a finite abelian group by $\mathbb{G}_{m,k}^g$. It is not hard to check that this finite group is isomorphic to $\mathbb{Z}^g/\{\mathrm{ord}(\lambda) \mid \lambda \in \Lambda\}$.

As a more concrete example of this, let's consider the analytic reduction of the Tate curve $\mathcal{E} = \mathbb{G}_{m,K}^{\mathrm{an}}/q^{\mathbb{Z}}$. Let π be a uniformizer of R, and let $\mathrm{ord}(q) = m > 1$. Take the affinoid covering $\mathbb{G}_{m,K}^{\mathrm{an}} = \bigcup_{n\in\mathbb{Z}} X_n$, where $X_n = \{x \in K \mid |\pi|^n \leq |x| \leq |\pi|^{n-1}\}$. Then $q^{\mathbb{Z}}$ acts properly discontinuously on this affinoid covering. Indeed, $qX_n \cap X_n = \emptyset$ since we assume $m > 1$. The analytic reduction $\overline{\mathcal{E}}$ is a union of m copies of $\mathbb{P}_k^1$, arranged as the sides of a polygon, and intersecting each other transversally.

4.4. **Formal schemes and their generic fibres.** There is a close, and very useful, relation between the formal schemes over R and rigid spaces over K. We will indicate this relation first making it explicit for the affinoid space $\mathrm{Sp}(T_n)$. Consider the polynomial ring $P = R[z_1, \dots, z_n]$. The formal completion of P with respect to the ideal πP is by definition the projective limit $\varprojlim P/\pi^m P$. This R-algebra is denoted by $\mathfrak{T}_n = R\langle z_1, \dots, z_n\rangle$. It consists of the power series $\sum_{\alpha\in\mathbb{N}^n} c_\alpha z^\alpha$ with all $c_\alpha \in R$ and such that for every $m > 0$ there are only finitely many α's with $c_\alpha \notin \pi^m R$. The affine formal scheme $\mathrm{Spf}(\mathfrak{T}_n)$ is the ringed space with underlying topological space $\mathrm{Spec}\ k[z_1, \dots, z_n]$, and structure sheaf $R\langle z_1, \dots, z_n\rangle$. The formal scheme $\mathrm{Spf}(\mathfrak{T}_n)$ can be seen as the completion of the affine scheme $\mathrm{Spec}(R[z_1, \dots, z_n])$ along its closed fibre $\pi = 0$. The connection to $\mathrm{Sp}(T_n)$ results from the isomorphism $K \otimes_R \mathfrak{T}_n = T_n$. Now we extend this relation to a functor from (certain) formal schemes over R to rigid spaces over K.

Let A be a reduced affinoid algebra. Consider the formal scheme $\mathrm{Spf}(A^\circ)$. There is a continuous surjective homomorphism of R-algebras $R\langle z_1, \dots, z_n\rangle \to A^\circ$ induced from the surjection of K-algebras $T_n \to A$. The underlying topological space of $\mathrm{Spf}(A^\circ)$ is the analytic reduction $\mathrm{Spec}(\overline{A})$, and $\mathrm{Sp}(K \otimes_R A^\circ) = \mathrm{Sp}(A)$ is the rigid space associated to $\mathrm{Spf}(A^\circ)$. This construction can be extended to formal schemes which are separated, reduced, locally of finite type and flat over R. One covers the formal scheme $\mathfrak{X}$ by formal affines $\mathrm{Spf}(\mathfrak{A})$, and glues $\mathrm{Sp}(\mathfrak{A} \otimes_R K)$. The resulting rigid analytic space is called the *generic fibre* of $\mathfrak{X}$, and is denoted by $\mathfrak{X}^{\mathrm{rig}}$. One can check that $\mathfrak{X} \to \mathfrak{X}^{\mathrm{rig}}$ is indeed a functor; see [BL93].

Example 4.10. Given a separated scheme X over R which admits a locally finite affine covering, there are two ways to associate a rigid analytic space to it. First, we could consider the generic fibre $X_K := X \times_R K$ of X, which is a locally finite type K-scheme, and take its analytification X_K^{an}. Second, we could consider the formal completion $\mathfrak{X}$ of X along its closed fibre (i.e., the formal completion of X with respect to an ideal of definition (π) of R), and then take its generic fibre $\mathfrak{X}^{\mathrm{rig}}$.

There is a morphism

$$i_X : \mathfrak{X}^{\mathrm{rig}} \to (X \times_R K)^{\mathrm{an}} \tag{4.1}$$

which is an open immersion in general, and an isomorphism for proper X over R; see [Con99, Thm. 5.3.1].

The most important example for us is when $X = \mathbb{G}_{m,R}$, a split torus over R. Then $(X \times_R K)^{\mathrm{an}} = \mathbb{G}_{m,K}^{\mathrm{an}}$ is the analytic one-dimensional torus over K and $\mathfrak{X} = \mathrm{Spf}(R\langle T, T^{-1}\rangle)$. Hence $\mathfrak{X}^{\mathrm{rig}} = \mathrm{Sp}(K\langle T, T^{-1}\rangle)$ is the unit circle. In this case i_X is the open immersion of the unit circle into the "punctured plane".

Finally, we recall Grothendieck's formal GAGA theorems [GD63, Ch.5], which we will need in §4.5:

Let X be a proper scheme over $\mathrm{Spec}(R)$. Denote the formal completion of X along π by $\widehat{X}$. Then $X \to \widehat{X}$ is a fully faithful functor. If $\mathfrak{X}$ is proper over $\mathrm{Spf}(R)$, and $\mathcal{L}$ is an invertible $\mathcal{O}_{\mathfrak{X}}$-module such that $\mathcal{L}_0 := \mathcal{L}/\pi$ is ample on $\mathfrak{X}_0 = \mathfrak{X} \times_{\mathrm{Spf}(R)} k$, then there is a proper scheme X over $\mathrm{Spec}(R)$ such that $\mathfrak{X} \cong \widehat{X}$.

4.5. **Abelian varieties which admit analytic uniformization.** We return to our initial question about exactly which abelian varieties over complete discrete valued fields are quotients of analytic tori by lattices.

Let $G = T^{\mathrm{an}}/\Lambda$, and let $\mathcal{U}$ be the explicit affinoid covering used to give G its analytic structure as in Example 4.9. As we discussed in §4.4, we can construct a formal scheme $\mathfrak{G}$, using the covering $\mathcal{U}$, such that $\mathfrak{G}^{\mathrm{rig}} \cong G$ (recall that G is proper). Assume G is algebraic. Then from Grothendieck's GAGA we also conclude that $\mathfrak{G}$ is algebraic. Let $\mathcal{G}$ be the proper scheme over $\mathrm{Spec}(R)$ such that $\widehat{\mathcal{G}} \cong \mathfrak{G}$. There is an isomorphism $\mathcal{G}_K^{\mathrm{an}} \cong G$; c.f. Example 4.10. Let A be the abelian variety for which $A^{\mathrm{an}} \cong G$. Then $\mathcal{G}$ is a model of A over R. Let $\mathcal{G}'$ be the scheme over R which is obtained by removing the singular locus of $\mathcal{G}_k$ from $\mathcal{G}$. Then $\mathcal{G}'$ is smooth, its closed fibre is an extension of a finite abelian group by $\mathbb{G}_{m,k}^g$; c.f. Example 4.9. Let $\mathcal{A}$ be the Néron model of A over R. By the universal property of Néron models the isomorphism $\mathcal{G}'_K \xrightarrow{\sim} \mathcal{A}_K$ uniquely extends to an R-morphism $\mathcal{G}' \to \mathcal{A}$. The image of $\mathcal{G}'$ is clearly open in $\mathcal{A}$. Hence the connected component of the identity $\mathcal{A}^0$ of $\mathcal{A}$ is isomorphic to the split algebraic torus $\mathbb{G}_{m,k}^g$. Abelian varieties whose Néron models have this property are called *totally degenerate*. We conclude that abelian varieties which admit analytic uniformization must be totally degenerate. (Being totally degenerate is a rather special property, for example, the abelian varieties with good reduction over R do not have this property.)

The converse is also true. Let A be a totally degenerate abelian variety of dimension g over K. Since $\mathcal{A}_K^0 = A$, on the right side of (4.1) we have the analytification A^{an} of A. On the other hand, $\mathcal{A}_k^0 \cong \mathbb{G}_{m,k}^g$. Then the lifting of tori in [Gro70, Exp. IX, Thm. 3.6] implies that the formal completion of $\mathcal{A}^0$ along its closed fibre is canonically isomorphic to a formal split torus $\widehat{\mathbb{G}}_m^g = (\mathrm{Spf}(R\langle T, T^{-1}\rangle))^g$. So we get an open immersion of analytic groups $i_{\mathcal{A}^0} : (\mathrm{Sp}(K\langle T, T^{-1}\rangle))^g \hookrightarrow A^{\mathrm{an}}$. We also have the analytic torus $G = (\mathbb{G}_{m,K}^{\mathrm{an}})^g$ associated to $(\widehat{\mathbb{G}}_m^g)^{\mathrm{rig}}$, and an open immersion $(\widehat{\mathbb{G}}_m^g)^{\mathrm{rig}} \hookrightarrow G$. The key fact is that $i_{\mathcal{A}^0}$ extends uniquely to a rigid analytic group morphism $G \to A^{\mathrm{an}}$, whose kernel is a lattice $\Lambda \subset G(K)$ of rank g, and we have an isomorphism of analytic groups $G/\Lambda \cong A^{\mathrm{an}}$; see [BL91, Thm. 1.2].

Remark 4.11. If one allows more general types of analytic uniformization then, in some sense, all abelian varieties over K can be uniformized. Consider the extensions of abelian varieties over K by split algebraic tori. Each such extension is an algebraic group G which sits in a short exact sequence

$$0 \to T \to G \to A \to 0,$$

where $T = \mathbb{G}_{m,K}^g$ and A an abelian variety. We will assume that A has good reduction over R, i.e., A extends to an abelian scheme over $\mathrm{Spec}(R)$. Let G^{an} be the analytification of G. One can define an appropriate notion of a rank-g lattice $\Lambda \subset G(K)$. It can be shown that G^{an}/Λ is a proper analytic space. Moreover, it is algebraic if a certain generalization of the Riemann form condition in Theorem 4.3 holds. The abelian variety B such that $B^{\mathrm{an}} \cong G^{\mathrm{an}}/\Lambda$ has Néron model $\mathcal{B}$ with $0 \to \mathbb{G}_{m,k}^g \to \mathcal{B}_k^0 \to \mathcal{A}_k \to 0$. In particular, B has semi-stable reduction. Conversely, every abelian variety with semi-stable reduction has a uniformization as above. We refer to [FvdP04, §6.7.2] (and the references therein) for the details.

It is known [Gro72, Thm. 3.6] that for every abelian variety B over K there is a finite galois extension K' of K such that $B_{K'}$ acquires a semi-stable reduction. Hence B can be uniformized over K'.

Remark 4.12. The pairing in Theorem 4.3 can be identified with Grothendieck's *monodromy pairing* in [Gro72]. Hence rigid geometry can be applied to the study of component groups of Néron models.

References

[BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-archimedean analysis. A systematic approach to rigid analytic geometry*, Springer, 1984.

[BL91] S. Bosch and W. Lütkebohmert, *Degenerating abelian varieties*, Topology **30** (1991), 653–698.

[BL93] ———, *Formal and rigid geometry I*, Math. Annalen **295** (1993), 291–317.

[Car90] H. Carayol, *Non-abelian Lubin-Tate theory*, in: Automorphic forms, Shimura varieties, and L-functions (1990), 15–39.

[CF90] C.-L. Chai and G. Faltings, *Degenerations of abelian varieties*, Springer, 1990.

[Con99] B. Conrad, *Irreducible components of rigid spaces*, Ann. Inst. Fourier **49** (1999), 473–541.

[Dri74] V. Drinfeld, *Elliptic modules*, Math. Sbornik **94** (1974), 594–627.

[FvdP04] J. Fresnel and M. van der Put, *Rigid analytic geometry and its applications*, Birkhäuser, 2004.

[GD63] A. Grothendieck and J. Dieudonné, *Étude cohomologique des faisceaux cohérents*, Publ. Math. IHES 11, 17, EGA III, 1961, 1963.

[Gro70] A. Grothendieck, *Groupes de type mulitplicatif: Homomorphismes dans un schema en groupes*, SGA 3, Exposé IX, 1970.

[Gro72] ———, *Modèles de Néron et monodromie*, SGA 7, Exposé IX, 1972.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Ann. Math. Studies 151, Princeton Univ. Press, 2001.

[LRS93] G. Laumon, M. Rapoport, and U. Stuhler, *$\mathcal{D}$-elliptic sheaves and the Langlands correspondence*, Invent. Math. **113** (1993), 217–338.

[Mum70] D. Mumford, *Abelian varieties*, Oxford University Press, 1970.

[Mum72a] ———, *An analytic construction of degenerating abelian varieties over complete local rings*, Compositio Math. **24** (1972), 239–272.

[Mum72b] ———, *An analytic construction of degenerating curves over complete local rings*, Compositio Math. **24** (1972), 129–174.

[OvdP88] F. Oort and M. van der Put, *A construction of an abelian variety with a given endomorphism algebra*, Compositio Math. **67** (1988), 103–120.

[Ray94] M. Raynaud, *Revêtements de la droite affine en charactéristique* $p > 0$ *et conjecture d'Abhyankar*, Invent. Math. **116** (1994), 425–462.

[Rib90] K. Ribet, *On the modular representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *arising from modular forms*, Invent. Math. **100** (1990), 431–476.

[Sil86] J. Silverman, *The arithmetic of elliptic curves*, GTM 106, Springer, 1986.

[Sil94] ______, *Advanced topics in the arithmetic of elliptic curves*, GTM 151, Springer, 1994.

[Tat71] J. Tate, *Rigid analytic spaces*, Invent. Math. **12** (1971), 257–289.

Department of Mathematics, Stanford University, Stanford, CA 94305
E-mail address: papikian@math.stanford.edu