THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The sixty-sixth annual William Lowell Putnam Mathematical Competition will take place on Saturday, December 1, from 8 to 11 and from 1 to 4. Around 4000 students will take it, and they will be among the best in the continent.

There will be six problems in each session, for a total of twelve. Each problem is worth 10 points, and there is very little partial credit. (The scores per problem are almost always 0, 1, 2, 8, 9, or 10. 8 is essentially correct with small gaps, and 2 is for very serious progress. So don't try to just get part marks on many problems, because you won't. Instead, you should try to figure out a problem, and then write it up very very well.) In a typical year, the median score will be 0 or 1 out of 120. So getting a point is a major accomplishment, and solving a problem even more so. Thus the Putnam is really a competition between you and the problems, not between you and other people.

Because these are hard problems, the strategy is different. The challenge is to sit down for three hours, look over a list of six problems, and try to figure one out and write it up. They are hard not because they have many parts, or have lots of computation; they solutions are very short, but ingenious. For sample questions, see the attached competition from 1988. They are all proof questions, meaning that you have to not just give an answer, but explain why it's true in a rigorous manner, not just beyond a reasonable doubt. In general they don't require much background, so freshmen are only at a slight disadvantage compared to upper years. Some sample problems are below, and you can see more on our webpage:

http://math.stanford.edu/∼vakil/putnam07/

One of the Putnam's idiosyncratic rules is that only people signed up well in advance are allowed to take it. But if you're signed up, you don't have to take it. So if there's a remote chance that you'll want to take it, please sign up; you're not committing yourself. I'll then e-mail you all later tonight, to find out which times and days of the week are bad for you. Then I'll book a room.

Why it's worth writing the Putnam.

- for the challenge
- a different kind of thinking than homework problems, much more akin to mathematical research
- it's worth seeing what these problems are like

Date: Monday, October 1, 2007.

- (can help in applying to math grad school)
- perhaps most important: the way of thinking you pick up will make understanding more advanced mathematical ideas that much easier

What you have to do.

(1) Sign up if you *might* take it! Name, e-mail address (you'll get e-mail from me soon). I have to submit Stanford's slate very soon (although a few additions are possible up until some time in November). If you end up being busy on December 2 and can't write, that's fine.
(2) Shortly before the Putnam, I'll e-mail you to say where it is; there will also be posters around the math department.
(3) *(Optional)* I will run a dinner-time problem-solving seminar. What we do will depend on who is there, but no background will be assumed. Usually: Half-hour on a technique, an hour of problems. We'll often have guest speakers, usually professors or post-docs who have done well on the Putnam or on the International Mathematical Olympiad.
(4) For more experienced people: I'll run a Masterclass once per week, immediately after the regular seminar.

How to prepare. Talk to me. Browse through Loren Larson's *Problem Solving through Problems*, or at old problems and solutions (e.g. in *The William Lowell Putnam Mathematical Competition 1985–2000: Problems, Solutions, and Commentary*); both books are on reserve at the library. See the website for more information too.

(This competition had more gettable questions than others in the last couple of decades. The median score was 16/120.)

A1. Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.

A2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

A3. Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^x$$

converges.

A4.(a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?
(b) What if "three" is replaced by "nine"?

Justify your answers.

A5. Prove that there exists a *unique* function f from the set $\mathbb{R}^+$ of positive real numbers to $\mathbb{R}^+$ such that

$$f(f(x)) = 6x - f(x) \qquad \text{and} \qquad f(x) > 0 \qquad \text{for all } x > 0.$$

A6. If a linear transformation A on an n-dimensional vector space has $n + 1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

B1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, and z positive integers.

B2. Prove or disprove: if x and y are real numbers with $y \ge 0$ and $y(y+1) \le (x+1)^2$, then $y(y-1) \le x^2$.

B3. For every n in the set $\mathbb{Z}^+ = \{1, 2, \dots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c + d = n$. Find, with proof, the smallest positive real number g with $r_n \le g$ for all $n \in \mathbb{Z}^+$.

B4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

B5. For positive integers n, let $\mathbb{M}_n$ be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of $\mathbb{M}_n$. (According to one definition, the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

One may note that

$$\mathbb{M}_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{M}_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

B6. Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t the number $at + b$ is a triangular number if and only if t is a triangular number.

(The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \dots\}$.)

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