POLYA PROBLEM-SOLVING SEMINAR WEEK 7: MISCELLANEOUS PROBLEMS, AND PROBLEM-SOLVING STRATEGY

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The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

The Problems.

1. Let $f(n)$ be the number of digits of a positive integer n (written in base 10). Give a simple formula for $f(2^n) + f(5^n)$.

2. Find a positive integer whose first digit is 1 and which has the property that if this digit is transferred to the end of the number, the number is tripled. (How many such numbers are there?)

3. A number of teams participate in a round robin tournament where each pair of teams plays a game, and there are no ties. Show that the sum of the number of wins of all the teams equals the sum of the number of losses. Show that the sum of the *squares* of the number of wins of each team equals the sum of the squares of the number of losses. (1965 Putnam)

4. In each of n houses on a straight street live one or more children. At what point should all the children meet so that the sum of the distances that they walk is as small as possible? (1950 Putnam)

5. Let $\mathbb{N}$ be the set of positive integers. Define f on n by $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(n)+1$. Can you give a short description of this function? (from Zeitz' book)

6. (Reid Barton) A deck of 50 cards contains two cards labeled n for each $n = 1, 2, ..., 25$. There are 25 people seated at a table, each holding two of the cards in this deck. Each

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minute every person passes the lower-numbered card of the two they are holding to the right. Prove that eventually someone has two cards of the same number.

7. Find two real numbers such that their sum is 2 and the sum of their 5th powers is 22. Give an *exact* answer. (Evan O'Dorney)

8. Suppose I have a large sphere, and I cut a cylindrical hole right through the center, and I'm left with a bracelet of height $2r$. What is its volume?

9. Let $a(n)$ be the number of entries in the nth row of Pascal's triangle that are 1 modulo 3, and let $b(n)$ be the number of entries that are 2 modulo 3. Show that $a(n) - b(n)$ is always a power of 2.

10. Show that the hypervolume of the n-dimensional sphere of radius r is

$$\frac{\pi^{n/2} r^n}{\frac{n}{2}!}$$

if n is even. (In fact this is true if n is odd, if you can figure out what $(1/2)!$ is...)

Problem of the week. Suppose that a rectangle is cut into finitely many smaller rectangles, with sides parallel to the larger rectangle's sides. Suppose also that each smaller rectangle has at least one side of integer length. Prove that the larger rectangle has at least one side of integer length.

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