# POLYA PROBLEM-SOLVING SEMINAR WEEK 2: NUMBER THEORY

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**The Rules.** These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on. If you would like to practice with the Pigeonhole Principle or Induction (a good idea if you haven't seen these ideas before), try those problems.

**The Hints.** Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

**The problems.**

**1.** Show that the sum of consecutive primes is never twice a prime.

**2.** Prove that $2^{70} + 3^{70}$ is divisible by 13. (Larson p. 99)

**3.** Find the smallest number with 28 divisors. (Larson p. 104)

**4.** Find the last two digits of $7^{7^{\cdot^{\cdot^{\cdot^{7}}}}}$ where the tower contains seven 7's.

**5.** Several positive integers are written on a chalk board. One can choose two of them, erase them, and replace them with their greatest common divisor and least common multiple. Prove that eventually the numbers on the board do not change.

**6.** Show that

$$n = \sum_{d|n} \phi(d)$$

where $\phi$ is Euler's function.

**7.** Prove that the only integer solution to the system

$$x^2 + 5y^2 = z^2$$
$$5x^2 + y^2 = t^2$$

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is the trivial solution $x = y = z = t = 0$. (*Putnam and Beyond* #707)

**8.** Show that
$$\sum_{i=1}^{n} \phi(i) \left\lfloor \frac{n}{i} \right\rfloor = \frac{n(n+1)}{2}.$$

**9.** Show that there exists an increasing sequence $(a_n)_{n\geq 1}$ of positive integers so that for any $k \geq 0$ the sequence $(k + a_n)_{n\geq 1}$ contains only finitely many primes. (*Putnam and Beyond* # 789)

**10.** Show that
$$1 + \frac{1}{2} + ... + \frac{1}{n}$$
is not an integer for any $n > 1$.

*This handout can be found at* **http://math.stanford.edu/~vakil/putnam07/**

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