# PROBLEM-SOLVING MASTERCLASS WEEK 6

**1.** Show that for any positive integer $n$, there is an integer $N$ such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n$$

where the $c_i$ are rational numbers and each $a_{ij}$ is one of the numbers $-1, 0, 1$. (Ryan Williams, 2004A4)

**2.** *(Problem-of-the-week)* There are a hundred prisoners numbered one to hundred, and a room with a hundred boxes numbered one to hundred. There are hundred slips of papers numbered 1 to 100; they are scrambled randomly, and one slip is placed inside each box.

Each prisoner is led into the room and is allowed to inspect the numbers in up to 50 boxes of his choice. If all the prisoners see their own number in one of the boxes that they inspect, then they are all set free.

The prisoners may initially agree on a strategy with which to open boxes, but after this no communication among them is allowed. Can their odds be better than $1/2^{100}$? (Leo Goldmakher)

**3.** For a positive real number $\alpha$, define

$$S(\alpha) = \{ \lfloor n\alpha \rfloor : n = 1, 2, 3, \ldots \}.$$

Prove that $\{1, 2, 3, \ldots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha), S(\beta)$ and $S(\gamma)$. (Olena Bormashenko, 1995B6)

**4.** There is a 12-sided polygon inscribed in a unit circle (radius = 1). If you multiply the lengths of all sides and all diagonals of this polygon what will be the result? (Meng-Hsuan Wu, from the PuzzleUp competition in 2006)

**5.** For which real numbers $c$ is there a straight line that intersects the curve

$$y = x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points? (Ravi Vakil, 1994B2)

*Date*: Monday, November 12, 2007.