PROBLEM-SOLVING MASTERCLASS WEEK 5

1. An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$. (Jackson Gorham, 2004A5)

2. For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k. (Kiat Chuan Tan, 1999B5)

3. *The "Lights-out" game, last week's problem-of-the-week.* Suppose $n \geq 2$ light bulbs are arranged in a row, numbered 1 through n. Under each bulb is a button. Pressing the button will change the state of the bulb above it (from on to off or vice versa), and will also change the state's neighbors. (Most bulbs have two neighbors, but the bulbs on the end have only one.) The bulbs start off randomly (some on and some off). For which n is it guaranteed to be possible that by flipping some switches, you can turn all the bulbs off? (Ravi Vakil)

4. There is a 12-sided polygon inscribed in a unit circle (radius = 1). If you multiply the lengths of all sides and all diagonals of this polygon what will be the result? (proposed by Meng-Hsuan Wu, from the PuzzleUp competition in 2006)

5. *Conway Checkers.* A checker is placed at every lattice point (x, y) with $y \leq 0$. A move consists of jumping one checker over an adjacent checker onto an empty lattice point, then removing the jumped checker. Prove that no checker can ever reach a point (x, y) with $y \geq 5$. (Bob Hough)

Date: Monday, November 5, 2007.