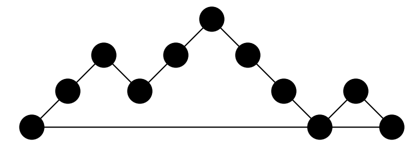
PROBLEM-SOLVING MASTERCLASS WEEK 3

1. A Dyck n-path is a lattice path of n upsteps $(1, 1)$ and n downsteps $(1, -1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n - 1)$-paths. (Olena Bormashenko, 2003A5)

2. Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that

a) Each contestant solved at most six problems, and
b) For each pair of a girl and a boy, there was at least one problem that both of them solved.

Prove that there is a problem that was solved by at least three girls and at least three boys. (Silas Johnson, IMO2001 #3)

3. An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$. (Jackson Gorham, 2004A5)

4. You have cards numbered from 1 to 20. First you will put them in order, and then you will shuffle them randomly. You will examine the distribution of cards before and after shuffling in order to determine the operation. Your aim is to get the initial order, repeating the operation. What is the maximum number of shuffles necessary in order to reach the initial order? (If the question was asked for four cards, the answer would be 4.) (Meng-Hsuan Wu, from the “puzzleup” weekly puzzle competition)

Date: Monday, October 22, 2007.

5. Show that for any positive integer n, there is an integer N such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1, 0, 1$. (Ryan Williams, 2004A4)

6. Consider a circle whose circumference is the golden mean $\tau = (1+\sqrt{5})/2$ (approx. 1.61803). Start at any point on the circle, and take some number of consecutive steps of arc length one in the clockwise direction. Number the points you step on in the order you encounter them, labeling your first step P_1, your second step P_2, and so on. When you stop, the difference in the subscripts of any two adjacent numbers is a Fibonacci number. (Ravi Vakil, *A Mathematical Mosaic (2nd ed., p. 163)*)