PROBLEM-SOLVING MASTERCLASS WEEK 2

1. Evaluate $\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$. (Ryan Williams, 2005A5)

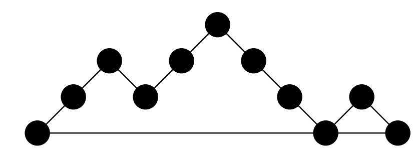
2. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \le m$. Show that $f(m,n) = f(n,m)$. (Kiat Chuan Tan, 2005B4)

3. Find all differentiable functions $f : (0,\infty) \to (0,\infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$. (George Margulis, 2005B3)

4. A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths. (Olena Bormashenko, 2003A5)

5. Consider a circle whose circumference is the golden mean $\tau = (1+\sqrt{5})/2$ (approx. 1.61803). Start at any point on the circle, and take some number of consecutive steps of arc length one in the clockwise direction. Number the points you step on in the order you encounter them, labeling your first step P_1, your second step P_2, and so on. When you stop, the difference in the subscripts of any two adjacent numbers is a Fibonacci number. (Ravi Vakil, *A Mathematical Mosaic (2nd ed., p. 163)*)

6. Show that for any positive integer n, there is an integer N such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1, 0, 1$. (Ryan Williams, 2004A4)

Date: Monday, October 15, 2007.