THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The sixty-sixth annual William Lowell Putnam Mathematical Competition will take place on Saturday, December 2, from 8 to 11 and from 1 to 4. Around 4000 students will take it, and they will be among the best in the continent.

There will be six problems in each session, for a total of twelve. Each problem is worth 10 points, and there is very little partial credit. (The scores per problem are almost always 0, 1, 2, 8, 9, or 10. 8 is essentially correct with small gaps, and 2 is for very serious progress. So don't try to just get part marks on many problems, because you won't. Instead, you should try to figure out a problem, and then write it up very very well.) In a typical year, the median score will be 0 or 1 out of 120. So getting a point is a major accomplishment, and solving a problem even more so. Thus the Putnam is really a competition between you and the problems, not between you and other people.

Because these are hard problems, the strategy is different. The challenge is to sit down for three hours, look over a list of six problems, and try to figure one out and write it up. They are hard not because they have many parts, or have lots of computation; they solutions are very short, but ingenious. For sample questions, see the attached competition from 1988. They are all proof questions, meaning that you have to not just give an answer, but explain why it's true in a rigorous manner, not just beyond a reasonable doubt. In general they don't require much background, so freshmen are only at a slight disadvantage compared to upper years. Some sample problems are below, and you can see more on our webpage:

http://math.stanford.edu/∼vakil/putnam06/

One of the Putnam's idiosyncratic rules is that only people signed up well in advance are allowed to take it. But if you're signed up, you don't have to take it. So if there's a remote chance that you'll want to take it, please sign up; you're not committing yourself. I'll then e-mail you all later tonight, to find out which times and days of the week are bad for you. Then I'll book a room.

Why it's worth writing the Putnam.

- for the challenge
- a different kind of thinking than homework problems, much more akin to mathematical research
- it's worth seeing what these problems are like

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- (can help in applying to math grad school)
- perhaps most important: the way of thinking you pick up will make understanding more advanced mathematical ideas that much easier

What you have to do.

(1) Sign up if you *might* take it! Name, e-mail address (you'll get e-mail from me soon). I have to submit Stanford's slate very soon (although a few additions are possible up until some time in November). If you end up being busy on December 2 and can't write, that's fine.
(2) Shortly before the Putnam, I'll e-mail you to say where it is; there will also be posters around the math department.
(3) *(Optional)* I will run a dinner-time problem-solving seminar. What we do will depend on who is there, but no background will be assumed. Usually: Half-hour on a technique, an hour of problems. We'll often have guest speakers, usually professors or post-docs who have done well on the Putnam or on the International Mathematical Olympiad.
(4) For more experienced people: I'll run a Masterclass once per week, immediately after the regular seminar.

How to prepare. Talk to me. Browse through Loren Larson's *Problem Solving through Problems*, or at old problems and solutions (e.g. in *The William Lowell Putnam Mathematical Competition 1985–2000: Problems, Solutions, and Commentary*); both books are on reserve at the library. See the website for more information too.

A1. Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

A2. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

A3. Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

A4. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

A5. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \ldots \right\}.$$

A6. Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

B1. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

B2. Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

> Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

B3. Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

B4. An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

B5. A palindrome in base b is a positive integer whose base-b digits read the same backwards and forwards; for example, 2002 is a 4-digit palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is the 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3-digit palindrome in base b for at least 2002 different values of b.

B6. Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients are congruent modulo p.)

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