PUTNAM PROBLEM-SOLVING SEMINAR WEEK 6: ALGEBRAIC TECHNIQUES

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The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Things to know. Move everything to one side. Factor. Expand. Complete the square! $x^n - y^n$, $x^{2m+1} + y^{2m+1}$, $x^4 + 4$, $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$. Trick for partial sums: $1/(x+1)(x+2)(x+3) = ?/(x+1) + ?/(x+2) + ?/(x+3)$. A degree n polynomial is determined by its values at $n + 1$ points. Newton sums (coefficients of a polynomial in terms of its roots); how to get sums of powers of roots. Rational roots theorem. Remainder theorem. Descartes rule of signs (proof using Rolle).

W1. 1/(x+1)(x+2)(x+3) = ?/(x+1) + ?/(x+2) + ?/(x+3).

W2. If $P(x)$ is a polynomial of degree n such that $P(k) = k/(k + 1)$ for $k = 0, \ldots, n$, determine $P(n + 1)$. (USAMO 1985)

W3. Show that four points on the parabola $y = x^2$, (a, a^2), ..., (d, d^2) (a, ..., d distinct) are concylic if and only if $a + b + c + d = 0$.

W4. Show that $\sqrt{2}$ is irrational.

1. Solve the system of equations

$$\begin{aligned}
2x_1 + x_2 + x_3 + x_4 + x_5 &= 6 \\
x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12 \\
x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24 \\
x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48 \\
x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96
\end{aligned}$$

Date: Monday, November 13, 2006.

2. (a) It is known that a quadratic equation has either 0, 1, or 2 unique real solutions. But consider the equation

$$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1$$

where a, b, and c are distinct. Notice that $x = a$, $x = b$, and $x = c$ are all solutions — how can this equation have three solutions? (From William Wu's wonderful website `http://www.ocf.berkeley.edu/~wwu/riddles/intro.shtml`) If you solve this problem, let me know, because I'll tell you an easy but powerful follow-up.

(b) Show, without multiplying out, that

$$\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c} = \frac{(a-b)(b-c)(a-c)}{abc}.$$

3. (a) Show that each number in the sequence 49, 4489, 444889, 44448889, ... is a perfect square. (Try not to use a calculator!)

(b) $[\sqrt{44}] = 6$, $[\sqrt{4444}] = 66$. Generalize and prove.

4. Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

5. *(The interpolation formula: important!)* Suppose a_1, ..., a_n are distinct numbers, and b_1, ..., b_n are given numbers, and $P(x)$ is a degree at most $n-1$ polynomial such that $P(a_i) = b_i$ for all i. Show that

$$P(x) = b_1 \frac{(x-a_2)(x-a_3)\cdots(x-a_n)}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)} + b_2 \frac{(x-a_1)(x-a_3)\cdots(x-a_n)}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)}$$

$$+ \cdots + b_n \frac{(x-a_1)(x-a_2)\cdots(x-a_{n-1})}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})}.$$

(Just ask I haven't written enough to make the pattern clear.)

6. Prove that $(2+\sqrt{5})^{1/3} + (2-\sqrt{5})^{1/3}$ is rational. (William Wu) (Ask me for a possible hint.)

7. Solve

$$(x^2 - 3x - 4)(x^2 - 5x + 6)(x^2 + 2x) + 30 = 0.$$

(Dragos Oprea)

8. Find a formula for

$$\left[\frac{n+2^0}{2^1}\right] + \left[\frac{n+2^1}{2^2}\right] + \left[\frac{n+2^2}{2^3}\right] + \cdots.$$

9. Work out $1/9899$ to 8 decimal places on your calculator. Explain.

10. The product of two of the four zeros of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Find k.

11. Show that the set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq 5/4$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

Sound's Problem of the Week. Look at the leading decimal digit of 2^n. Does 7 or 8 appear more often?

This handout can be found at **http://math.stanford.edu/~vakil/putnam06/**

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