PUTNAM PROBLEM-SOLVING SEMINAR WEEK 4: NUMBER THEORY

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The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Things to remember.

Divisibility and primes: Division algorithm, greatest common divisor and Euclid's algorithm, r and s relatively prime means that there are a and b such that $ar + bs = 1$, unique factorization, counting divisors of n, Euler ϕ-function $\phi(n) = \#$ numbers between 1 and n relatively prime to n. $\phi(n) = n(1 - 1/p_1) \cdots (1 - 1/p_k)$ where $p_1, \ldots, p_k$ are the distinct prime factors of n. Classic proofs: infinitely many primes, $\sqrt{2}$ is irrational, arguments using descent.

Mod arithmetic: Working with congruences, applications to positional (base-b) notation, Chinese Remainder Theorem, Wilson's Theorem $(p-1)! \equiv -1 \pmod{p}$, Fermat's little theorem $a^p \equiv a \pmod{p}$, Euler's generalization $a^{\phi(n)} \equiv 1 \pmod{n}$ if $\gcd(a, n) = 1$, some consequences for k-th roots/powers modulo n.

THE PROBLEMS:

W1. Why does p divide $\binom{p}{a}$, for p prime and $0 < a < p$? How many zeros does 2006! end with? How many zeros does $\binom{15}{7}$ end with?

W2. Use the fact that $10 \equiv -1 \pmod{11}$ to (re-)discover a rule for divisibility by 11. Can anything clever be said for divisibility by 7? (There may not be a unique answer!)

W3. For what a does the equation $x^2 \equiv a \pmod{7}$ have solutions? What about modulo 8? mod 9?

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W4. Find the smallest n such that $\phi(n) = 4$. (How many other such n are there?) Show that the equation $\phi(n) = 14$ has no solutions. Compute the (unique!) fifth root of 2 modulo 13.

1. Prove that consecutive Fibonacci numbers are always relatively prime.

2. Is it possible for 4 consecutive integers to be composite? How about 5? Arbitrarily many?

3. Suppose the number $n = 47^{99}$ is written in base ten. What is the last (rightmost) digit of n? How about the last *two* digits of n?

4. (a) Show that $n^7 - n$ is divisible by 42 for every positive integer n.
(b) Show that every prime not equal to 2 or 5 divides infinitely many of the numbers 1, 11, 111, 1111, etc.

5. (a) Find the largest prime less than 100 that divides $\binom{200}{100}$.
(b) Find the largest prime p such that p^2 divides $\binom{2000}{1000}$.

6. Show that there exist three consecutive integers, each of which is divisible by a 2006th power of an integer (not including 1^{2006} and $(-1)^{2006}$).

7. Let $a_n = 100 + n^2$ for $n \geq 1$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

8. Let $f(n)$ denote the sum of the digits of n (when n is expressed in base 10).
(a) For any integer n, prove that eventually the sequence $f(n), f(f(n)), f(f(f(n))), \ldots$ will become constant. Call this constant value the *digital root* of n.
(b) Prove that the digital root of the product of any two twin primes (i.e., primes that differ by 2), other than 3 and 5, is 8. (Hint: how does this relate to a question in W3?)
(c) (IMO 1975) Find $f(f(f(4444^{4444})))$ without a calculator.

9. Let a and b be positive integers, and define numbers x_n by $x_0 = 1$ and $x_{n+1} = ax_n + b$ for all $n \geq 0$. Prove that for any choice of a and b, the sequence $(x_n)_{n\geq 0}$ contains infinitely many composite numbers.

10. Suppose $(a_i)_{i\geq 1}$ is a sequence of positive integers satisfying $\gcd(a_i, a_j) = \gcd(i, j)$ for $i \neq j$. Show that $a_i = i$ for each i.

11. Prove that the number $\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$ is not divisible by 5 for any integer $n \geq 0$.

Extra Problem from Sound: For any natural number n let S_n denote the set of all numbers m such that the fractional part of n/m is at least $1/2$. Prove that the sum of $\phi(m)$ over all elements of S_n equals n^2.

This handout can be found at **http://math.stanford.edu/~vakil/putnam06/**

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