PUTNAM PROBLEM-SOLVING SEMINAR WEEK 2: GENERATING FUNCTIONS

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The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

THE PROBLEMS:

Warm-up 1. (a) Show that $\sum_{i=1}^{n} \binom{n}{i} = 2^n$. (b) Show that $\sum_{i=1}^{n} (-1)^i \binom{n}{i} = 0$ if $n > 0$.

Warm-up 2. Give a short formula for $\sum_{i=1}^{n} i \binom{n}{i}$.

Warm-up 3. Show that for any positive integer n,

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Warm-up 4. (a) Find a closed-form formula for A_n, where $A_0 = 2$, $A_1 = 5$, $A_n = 5A_{n-1} - 6A_{n-2}$. (b) Find a closed-form formula for B_n where $B_0 = 1$, $B_1 = 3$, $B_n = 4B_{n-1} - 4B_{n-2}$.

1. Give a formula for $\sum_{i=1}^{n} i^2 \binom{n}{i}$ where n is a non-negative integer.

2. (Vandermonde identity) Prove that for any positive integers $k < m, n$,

$$\sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} = \binom{n+m}{k}.$$

This is a handy fact, and the method of proof is handier still. (Here is a seemingly related problem from Dragos Oprea: if $d_1, \dots, d_k$ are fixed and $a_1, \dots, a_k$ vary but sum to a, find $\prod \binom{d_i}{a_i}$.)

3. Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Interpret the x^n term of the identity

$$e^x \cdot e^x = e^{2x}.$$

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4. Calculate $\sum_{i=0}^{\infty} \frac{i^2}{i!}$.

5. Let n be any positive integer. Show that the set of weights

$$1, 3, 3^2, 3^3, 3^4, \ldots$$

grams can be used to weight an n-gram weight (using both pans of a scale), and that this can be done in exactly *one* way. Explain this using generating functions.

6. Show that for each positive integer n, the number of partitions of n into unequal parts is equal to the number of partitions of n into odd parts. For example, if $n = 6$, there are 4 partitions into unequal parts, namely

$$1+5, \quad 1+2+3, \quad 2+4, \quad 6.$$

And there are also 4 partitions into odd parts,

$$1+1+1+1+1+1, \quad 1+1+1+3, \quad 1+5, \quad 3+3.$$

7. *(Important fact!!)* Suppose that in base p, $n = n_0 + n_1 p + \cdots + n_k p^k$ and $a = a_0 + a_1 p + \cdots + a_k p^k$. Show that

$$\binom{n}{a} \equiv \prod_{i=1}^{k} \binom{n_i}{a_i} \pmod{p}$$

(Possible application: how many odd binomial coefficients are there in the 2006th row of Pascal's triangle?)

8. Show that

$$\sum_{i=0}^{s} \binom{r+i}{r} \binom{t-r-i}{t-r-s} = \binom{t+1}{t-s+1}.$$

9. A finite sequence $a_1, a_2, \ldots, a_n$ is called p-*balanced* if any sum of the form $a_k + a_{k+p} + a_{k+2p} + \cdots$ is the same for any $k = 1, 2, \ldots, p$. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal to zero.

10. For nonnegative integers n and k, define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n, k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j}.$$

Extra problem from Sound: (a) Find

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots.$$

(b) Find

$$\frac{1}{1} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{11} - \cdots.$$

These series arise in number theory (they are L-values). The answer to (b) involves one of the most famous irrational numbers.

This handout can be found at **http://math.stanford.edu/˜vakil/putnam06/**

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