PUTNAM PROBLEM-SOLVING SEMINAR WEEK 1: INDUCTION AND PIGEONHOLE

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The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

THE PROBLEMS:

W1. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all positive integers n.

W2. Prove that there are two people in the U.S. right now with the same amount of hair on their heads (not including bald people!).

W3. Prove that in any group of six people there are either three mutual friends or three mutual strangers. (Possible follow-up: Find some n so that in any group of n people there are either four mutual friends or four mutual strangers.)

W4. Five points lie in an equilateral triangle of size 1. Show that two of the points lie no farther than 1/2 apart. Can the "1/2" be replaced by anything smaller? Can it be improved if the "five" is replaced by "six"?

1. Prove that all even perfect squares are divisible by 4. Prove that all odd perfect squares leave a remainder of 1 upon division by 8. (This is a useful fact to know!) What are the possible remainders when you divide a perfect square by 3?

2. (a) Prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

(b) Prove that

$$2!4!\cdots(2n)! \geq ((n+1)!)^n.$$

(Larson 2.1.6)

3. In how many ways can a $2 \times n$ rectangle be tiled with 2×1 dominoes? Give your answer in the form of a well-known function.

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4. Recall that the Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Show that

$$\sum_{n=2}^{\infty} \arctan\left(\frac{(-1)^n}{F_{2n}}\right) = \frac{1}{2}\arctan\frac{1}{2}.$$

(Hint: What is this problem doing on this list of problems? Follow-up hint: This is probably not a pigeonhole principle problem!)

5. Show that if there are n people at a party, then two of them know the same number of people (among those present).

6. A group of n people play a round-robin arm-wrestling tournament. Each match ends in either a win or a lost. Show that it is possible to label the players $P_1, P_2, P_3, \ldots, P_n$ in such a way that P_1 defeated P_2, P_2 defeated $P_3, \ldots, P_{n-1}$ defeated P_n. (Larson 2.1.9)

7. A lattice point in the plane is a point (x, y) such that both x and y are integers. Find the smallest number n such that given n lattice points in the plane, there exist two whose midpoint is also a lattice point.

8. A polygon in the plane has area 1.2432. Show that it contains two distinct points (x_1, y_1) and (x_2, y_2) that differ by (a, b), where a and b are integers.

9. What is the fourth digit of $\sum_{i=0}^{[2006/3]} \binom{2006}{3i}$ — in binary?

10. If n is even, prove that the volume of an n-dimensional hypersphere of radius r is

$$\frac{\pi^{n/2} r^n}{(n/2)!}.$$

11. Let u be an irrational real number. Let S be the set of all real numbers of the form $a + bu$, where a and b are integers. Show that S is dense in the real numbers, i.e. for any real number x and any $\epsilon > 0$, there is an element $y \in S$ such that $|x - y| < \epsilon$. (Hint: first let $x = 0$.)

Extra problem from Sound: Does there exist a circle with radius 100 in the plane containing exactly 31415 lattice points (points of the form (a, b) where a and b are integers)?

This handout can be found at **http://math.stanford.edu/~vakil/putnam06/**

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