PROBLEM-SOLVING MASTERCLASS WEEK 7

1. Find the smallest term in the sequence $\{a_n\}$ where $a_1 = 1993^{1994^{1995}}$, $a_{n+1} = a_n/2$ if a_n is even, and $a_n + 7$ if a_n is odd. (Dragos Oprea)

2. For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy $(x^2 + y^2)^m = (xy)^n$. (Kiyoto Tamura, 1992A3)

3. Choose n arbitrary points $A_1, A_2, \ldots, A_n$ on the circumference of a circle ($n > 1$). Let M be their geometric center of mass. Extend each line segment A_iM to its second point of intersection with the circle and call these points B_i for each i. Prove that the sum over all i of the A_iM/MB_i is equal to n, where A_iM is the length of the line segment A_iM and MB_i is the length of the line segment MB_i. (Boris Hanin)

4. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix. (Jackson Gorham, 1985B6)

5. Let ABC be a triangle. Points D, E, F are on sides BC, CA, AB respectively, such that $DC + CE = EA + AF = FB + BD$. Prove that

$$DE + EF + DF \geq \frac{AB + BC + CA}{2}.$$

6. For any natural number n, let S_n denote the set of all numbers m such that the fractional part of n/m is at least $1/2$. Prove that the sum of $\phi(m)$ over all elements of S_n equals n^2. (Mark Lucianovic; Sound's problem heard from Terry Tao)

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