PROBLEM-SOLVING MASTERCLASS WEEK 2

1. Triangle ABC has an area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST. (Nathan Pflueger, 2001A4)

2. Alberto places N checkers in a circle. Some, perhaps all, are black; the others are white. Betul places new checkers between the pairs of adjacent checkers in Alberto's ring: she places a white checker between every two that are the same color and a black checker between every pair of opposite color. She then removes Alberto's original checkers to leave a new ring of N checkers in a circle. Alberto then performs the same operation on Betul's ring of checkers following the same rules. The two players alternately perform this maneuver over and over again. Show that if N is a power of two, then all the checkers will eventually be white, no matter the arrangement of colors Alberto initially puts down. (Ravi Vakil, from Zeitz' *The Art and Craft of Problem-Solving*, p. 142.)

3. *(of friends and politicians)* Suppose in a group of people we have the situation that any pair of persons have precisely one common friend. Then there is necessarily a "politician" who knows everyone. (Theo Johnson-Freyd, from *Proofs from the Book*)

4. For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

(Nathan Pflueger, 2001B3)

5. Let $B(n)$ be the number of ones in the base two expression for the positive integer n. For example, $B(6) = B(110_2) = 2$ and $B(15) = B(1111_2) = 4$. Determine whether or not

$$e^{\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)}}$$

is a rational number. (Ravi Vakil, 1981B5)

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