THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The sixty-sixth annual William Lowell Putnam Mathematical Competition will take place on Saturday, December 3, from 8 to 11 and from 1 to 4. Around 3700 students will take it, and they will be among the best in the continent.

There will be six problems in each session, for a total of twelve. Each problem is worth 10 points, and there is very little partial credit. (The scores per problem are almost always 0, 1, 2, 8, 9, or 10. 8 is essentially correct with small gaps, and 2 is for very serious progress. So don't try to just get part marks on many problems, because you won't. Instead, you should try to figure out a problem, and then write it up very very well.) In a typical year, the median score will be 0 or 1 out of 120. So getting a point is a major accomplishment, and solving a problem even more so. Thus the Putnam is really a competition between you and the problems, not between you and other people.

Because these are hard problems, the strategy is different. The challenge is to sit down for three hours, look over a list of six problems, and try to figure one out and write it up. They are hard not because they have many parts, or have lots of computation; they solutions are very short, but ingenious. For sample questions, see the attached competition from 1988. They are all proof questions, meaning that you have to not just give an answer, but explain why it's true in a rigorous manner, not just beyond a reasonable doubt. In general they don't require much background, so freshmen are only at a slight disadvantage compared to upper years. Some sample problems are below, and you can see more on our webpage:

http://math.stanford.edu/∼vakil/putnam05/

One of the Putnam's idiosyncratic rules is that only people signed up well in advance are allowed to take it. But if you're signed up, you don't have to take it. So if there's a remote chance that you'll want to take it, please sign up; you're not committing yourself. I'll then e-mail you all later tonight, to find out which times and days of the week are bad for you. Then I'll book a room.

Why it's worth writing the Putnam.

- for the challenge
- a different kind of thinking than homework problems, much more akin to mathematical research
- it's worth seeing what these problems are like

Date: Monday, October 3, 2005.

- (can help in applying to math grad school)
- perhaps most important: the way of thinking you pick up will make understanding more advanced mathematical ideas that much easier

What you have to do.

(1) Sign up if you *might* take it! Name, e-mail address (you'll get e-mail from me soon). I have to submit Stanford's slate very soon (although a few additions are possible up until some time in November). If you end up being busy on December 3 and can't write, that's fine.

(2) Shortly before the Putnam, I'll e-mail you to say where it is; there will also be posters around the math department.

(3) *(Optional)* I will run a dinner-time problem-solving seminar. What we do will depend on who is there, but no background will be assumed. Usually: Half-hour on a technique, an hour of problems. We'll often have guest speakers, usually professors or post-docs who have done well on the Putnam or on the International Mathematical Olympiad. *New this year:* For those of you who would like to discuss the problems with each other during the week, Paul-Oliver Dehaye has set up a discussion group on Coursework. It is a bit counterintuitive to figure out how to join, but instructions are given on the Stanford Putnam webpage. If they don't work, let me know.

(4) For more experienced people: I'll run a Masterclass once per week, likely immediately after the regular seminar.

How to prepare. Talk to me. Browse through Loren Larson's *Problem Solving through Problems*, or at old problems and solutions (e.g. in *The William Lowell Putnam Mathematical Competition 1985–2000: Problems, Solutions, and Commentary*); both books are on reserve at the library. See the website for more information too.

1. The 1986 competition

(This competition had more gettable questions than others in the last couple of decades. The median score was 19/120.)

A1. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

A2. What is the units (i.e., rightmost) digit of $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$? Here $\lfloor x \rfloor$ is the greatest integer $\leq x$.

A3. Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2+n+1)$, where $\operatorname{Arccot} t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot \theta = t$.

A4. A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.
(b) The sum of the n entries of a transversal is the same for all transversals of A.

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

where the a_i's and b_i's are rational numbers.

A5. Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of n real variables $x = (x_1, \ldots, x_n)$ with continuous second-order partial derivatives everywhere on $\mathbb{R}^n$. Suppose further that there are constants c_{ij} such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all i and j, $1 \leq i \leq n$, $1 \leq j \leq n$. Prove that there is a function $g(x)$ on $\mathbb{R}^n$ such that $f_i + \partial g / \partial x_i$ is linear for all i, $1 \leq i \leq n$. (A linear function is one of the form

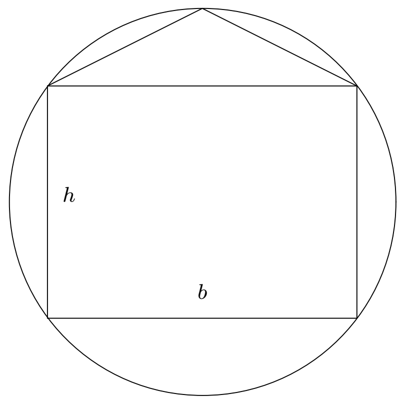
$$a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.)$$

A6. Let $a_1, a_2, \ldots, a_n$ be real numbers, and let $b_1, b_2, \ldots, b_n$ be distinct positive integers. Suppose there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \ldots, b_n$ and n (but independent of $a_1, a_2, \ldots, a_n$).

B1. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area?



B2. Prove that there are only a finite number of possibilities for the ordered triple $T = (x-y, y-z, z-x)$, where x, y, and z are complex numbers satisfying the simultaneous equations

$$x(x-1) + 2yz = y(y-1) + 2zx = z(z-1) + 2xy,$$

and list all such triples T.

B3. Let Γ consist of all polynomials in x with integer coefficients. For f and g in Γ and m a positive integer, let $f \equiv g \pmod{m}$ mean that every coefficient of $f - g$ is an integral multiple of m. Let n and p be positive integers with p prime. Given that f, g, h, r, and s are in Γ with $rf + sg \equiv 1 \pmod{p}$ and $fg \equiv h \pmod{p}$, prove that there exist F and G in Γ with $F \equiv f \pmod{p}$, $G \equiv g \pmod{p}$, and $FG \equiv h \pmod{p^n}$.

B4. For a positive real number r, let $G(r)$ be the minimum value of $\left|r - \sqrt{m^2 + 2n^2}\right|$ for all integers m and n. Prove or disprove the assertion that $\lim_{r\to\infty} G(r)$ exists and equals 0.

B5. Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. Let $p(x, y, z)$, $q(x, y, z)$, $r(x, y, z)$ be polynomials with real coefficients satisfying

$$f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).$$

Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x, \pm y, \pm z$, where the number of minus signs is 0 or 2.

B6. Suppose A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Here I is the $n \times n$ identity matrix, and if M is an $n \times n$ matrix, M^t is the transpose of M. Prove that $A^tD - C^tB = I$.

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