PUTNAM PROBLEM-SOLVING SEMINAR WEEK 7: PROBLEMS PROBLEMS PROBLEMS

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Things to remember for the Putnam.

What to bring: You should bring pencils, pens, erasers, and a watch. (There is no clock in the room, so the watch is important.) You're not allowed to bring any paper, books, slide rules, rulers, calculators, computers, etc. Fortunately, some of the infinite supply of paper is still left, so I'll bring that. (You are allowed to hand in solutions on these as well.) The honor code will be in effect.

Here are a few general tips:

- Wear comfortable clothes. If possible, get a good night's sleep beforehand.
- Spend some time at the start looking over the questions. The earlier questions in each half tend to be easier, but this isn't always the case.
- When your mind gets tired, take a break; go outside, get a snack, use the bathroom, clear your head.
- The "fifteen minute rule" can be helpful: if you find that you've been thinking about a problem aimlessly without having any serious new ideas for 15 minutes, then jump to a different problem, or take a break.
- Don't become discouraged. Don't think of this as a test or exam. Often a problem will break open a couple of hours into the session. And the problem with your name on it might be in the afternoon session. Remember that getting somewhere on one question is serious; getting a full question is a big deal.
- If you solve a problem, write it up very well (rather than starting a new problem). Grading on the Putnam is incredibly harsh.

And a few mathematical tips. Here are some things that people tend not to do for some reason (as I've observed in the Monday night seminars) even though they can be a big help.

- Try a few small cases out. Try a lot of cases out. (Three hours is a long time.)

Date: Monday, November 28, 2005.

- Use lots and lots of paper. It can help free your mind.
- Don't be afraid of diving into algebra. (Three hours is a long time.) You won't waste that much time, thanks to the 15-minute rule.
- If the question asks if something is true, and you have a guess that doesn't go anywhere, then try the other possibility.
- Try (seemingly) stupid things.
- Look for symmetries. Try to connect the problem to one you've seen before. Ask yourself "how would [person] solve this problem"?
- "The only way this problem could have a nice solution is if this particular approach worked out, unlikely as it seems, so I'll try it out."
- "I have no idea what to do on this problem, but I could imagine person X at the blackboard, who would try the following idea..."
- Show no fear. If you think a problem is probably too hard for you, but you have an idea, try it anyway. (Three hours is a long time.)

1. Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
and
(ii)$A_1 \cap A_2 \cap A_3 = \emptyset$,

where $\emptyset$ denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, and d are nonnegative integers. (1985A1)

2. Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$. (1985A3)

3. Let k be the smallest positive integer with the following property:

> There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial
>
> $$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$
>
> has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved. (1985B1)

4. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

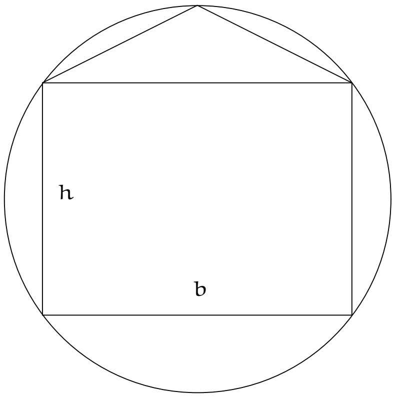
$$\frac{d}{dx}(f_{n+1}(x)) = (n+1)f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes. (1985B2)

5. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$. (1986A1)

6. What is the units (i.e., rightmost) digit of $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$? Here $\lfloor x \rfloor$ is the greatest integer $\leq x$. (1986A2)

7. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area? (1986B1)



8. The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the $10^{n\text{th}}$ digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100^{th} digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$. (1987A2)

9. Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area. (1988A1)

10. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \ldots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, and z positive integers. (1988B1)

11. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y + 1) \leq (x + 1)^2$, then $y(y - 1) \leq x^2$. (1988B2)

This handout can be found at **http://math.stanford.edu/˜vakil/putnam05/**

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