PUTNAM PROBLEM-SOLVING SEMINAR WEEK 6: CALCULUS

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Things to remember.

Convergence. The Dominated Convergence Theorem. The Monotone Convergence Theorem. Limit Comparison Test. Integral Comparison Test.

Continuity. The Intermediate Value Theorem. The Extreme Value Theorem. (More generally, a continuous function on a compact set attains its sup and inf.)

"Big O and little o notation": $O(g(n))$ is a stand-in for a function $f(n)$ for which there exists a constant C such that $|f(n)| \leq C|g(n)|$ for all sufficiently large n. (This does not necessarily imply that $\lim_{n \to \infty} f(n)/g(n)$ exists.) Similarly "$f(t) = O(g(t))$ as $t \to 0$" means that there exists a constant C such that $|f(t)| \leq C|g(t)|$ for sufficiently small nonzero t. $o(g(n))$ is a stand-in for a function $f(n)$ such that $\lim_{n \to \infty} f(n)/g(n) = 0$. One can similarly define "$f(t) = o(g(t))$ as $t \to 0$".

Calculus. Riemann Sums: if a function is Riemann-integrable, e.g. if it is continuous on a closed finite interval, then the integral is the limit of the Riemann sums.

Rolle's Theorem: Let $[a, b]$ be a closed interval in $\mathbb{R}$. Let $f(t)$ be a function that is continuous on $[a, b]$ and differentiable on (a, b), and suppose that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Inequalities of integrals: $f \leq g$ means $\int_a^b f \leq \int_a^b g$ if $a \leq b$.

Taylor's Formula with Remainder: if h has continuous nth derivatives, then for any $x > 0$ and integer $n > 0$, there exists $\theta_n \in [0, x]$ such that

$$h(x) = h(0) + h'(0)x + \cdots + h^{(n-1)}(0)x^{n-1}/(n-1)! + h^{(n)}(\theta_n)x^n/n!.$$

Mean Value Theorem for integrals: If f is continuous on $[a, b]$, then for some c in $[a, b]$ we have $\int_a^b f(x)dx = f(c)(b - a)$. For derivatives: If f is continuous on $[a, b]$ and has a derivative at each point of (a, b), then there is a point c of (a, b) for which $f(b) - f(a) = f'(c)(b - a)$.

1. Recall integration by parts:

$$\int f\, dg = fg - \int g\, df.$$

Substitute $f(x) = 1/x$, $g(x) = x$, and manipulate, to get

$$\int \frac{1}{x}\, dx = 1 + \int \frac{1}{x}\, dx.$$

Hence $0 = 1$. What has gone wrong?

2. Show that right now, there are two diametrically-opposed points on the earth's equator that are exactly the same temperature. (Follow-up problems: Are there two points on the earth's equator separated by 120 degrees that are exactly the same temperature? How about π degrees? Are there two diametrically opposed points on the earth's surface with the same temperature and air pressure?)

3. Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown:
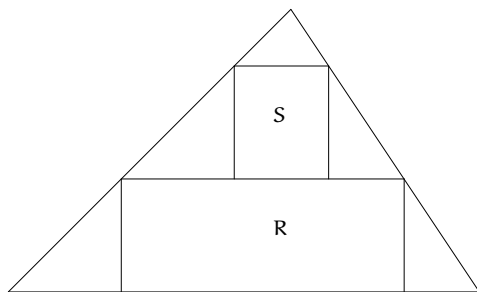
Let $A(X)$ denote the area of polygon X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above. (Putnam 1985A2)

4. Evaluate $\int_0^a \int_0^b e^{\max\{b^2 x^2, a^2 y^2\}}\, dy\, dx$, where a and b are positive. (Putnam 1989A2)

5. Find all real-valued continuously differentiable functions f on the real line such that for all x

$$(f(x))^2 = \int_0^x \left((f(t))^2 + (f'(t))^2\right)\, dt + 1990.$$

(Putnam 1990B1)

6. Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\, dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

7. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \le a_{2n} + a_{2n+1}$ for all $n \ge 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges. (Putnam 1994A1)

8. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots? (Putnam 1990B5)

9. For any pair (x, y) of real numbers, a sequence $(a_n(x, y))_{n \ge 0}$ is defined as follows:

$$a_0(x, y) = x,$$
$$a_{n+1}(x, y) = \frac{(a_n(x, y))^2 + y^2}{2}, \qquad \text{for } n \ge 0.$$

Find the area of the region $\{(x, y) | (a_n(x, y))_{n \ge 0} \text{ converges}\}$. (Putnam 1992B3)

10. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2 + 1}, \qquad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \ldots$. (Putnam 1992A4)

11. Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1\right)^x$$

converges. (Putnam 1988A3)

This handout can be found at **http://math.stanford.edu/˜vakil/putnam05/**

E-mail address: `vakil@math.stanford.edu`