# PUTNAM PROBLEM-SOLVING SEMINAR WEEK 5: INEQUALITIES

**The Rules.** These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

**The Hints.** Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

**Things to remember.** Sums of squares are non-negative. The arithmetic mean-geometric mean inequality (AM-GM): $\sum_{i=1}^{n} a_i \geq (\prod a_i)^{1/n}$ if the $a_i$ are non-negative. The triangle inequality (the shortest distance between two points is a straight line). Cauchy's inequality $(a_1b_1 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$. Lagrange multipliers.

**1.** (a) Find, without using calculus, the minimum surface area of a rectangular box which holds volume $V$. (Mark Lucianovic)
(b) Find, without using calculus, the minimum area of an open box — with no top — which holds volume $V$.

**2.** Determine the maximum value of $(\sin A_1)(\sin A_2)\cdots(\sin A_n)$ given that $(\tan A_1)(\tan A_2)\cdots(\tan A_n) =$ 1. (Hint: *don't* use calculus without thinking!) (Mark Lucianovic) [We discovered that this is hard for a # 2!]

**3.** Given $a, b, c$ nonnegative real numbers such that $(a+1)(b+1)(c+1) \leq 8$. Prove that $abc \leq 1$. (Mark Lucianovic)

**4.** If $x_1, \ldots, x_n$ are positive real numbers whose product is 1, show that $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} {x_i}^2$. (Soren Galatius)

**5.** The polynomial $4x^4 - ax^3 + bx^2 - cx + 5$ has four positive (real) roots such that $r_1/2 + r_2/4 + r_3/5 + r_4/8 = 1$. Find them. (Mark Lucianovic)

**6.** Suppose $a, b, c$ are positive real numbers. Prove that

$$\sqrt[3]{abc} + 1 \leq \sqrt[3]{(a+1)(b+1)(c+1)} \leq \left(\frac{\sqrt[3]{a+1} + \sqrt[3]{b+1} + \sqrt[3]{c+1}}{3}\right)^3 \leq \frac{a+b+c}{3} + 1.$$

(Soren Galatius)

*Date*: Monday, November 7, 2005.

**7.** Suppose $x_1, \dots, x_n$ are positive real numbers. Prove that

$$\frac{{x_1}^2}{x_1 + x_2} + \frac{{x_2}^2}{x_2 + x_3} + \cdots + \frac{{x_n}^2}{x_n + x_1} \geq \frac{x_1 + x_2 + \cdots + x_n}{2}.$$

Hint: Don't be afraid to introduce square roots. (Soren Galatius)

**8.** (A number-theoretic inequality) Given integers $0 < a < b < c < d < e$, prove that $1/\operatorname{lcm}(a,b) + 1/\operatorname{lcm}(b,c) + 1/\operatorname{lcm}(c,d) + 1/\operatorname{lcm}(d,e) \leq 15/16$. Generalize to integers $0 < a_1 < a_2 < \cdots < a_k$. (Mark Lucianovic)

**9.** *(The Power Mean inequality)* Suppose $a_1, \dots, a_n$ are positive real numbers. Define the $k$th power mean as

$$M_k := \left( \frac{a_1^k + \cdots + a_n^k}{n} \right)^{1/k}$$

if $k \neq 0$, and $M_0 = (a_1 \cdots a_n)^{1/n}$. Show that if $a > b$, then $M_a \geq M_b$, with equality if and only if all the $a_i$'s are equal. (This can be very handy! If $k = 1$, we get the arithmetic mean; if $k = 0$ we get the geometric mean; if $k = -1$ we get the *harmonic mean*; if $k = 2$ we get the *quadratic mean*.)

**10.** Prove the "logarithmic mean" inequality for $a > b > 0$:

$$\sqrt{ab} < \frac{a - b}{\ln a - \ln b} < \frac{a + b}{2}.$$

(Mark Lucianovic)

**11.** Suppose $f(x)$ is a continuous function $[a, b] \to \mathbb{R}^+$ $(a < b)$. Figure out what the integral version of the arithmetic mean should be (see the Power Mean statement above). State the integral version of the quadratic mean-arithmetic mean inequality. Prove it!

*This handout can be found at* **http://math.stanford.edu/~vakil/putnam05/**

*E-mail address*: `vakil@math.stanford.edu`