PUTNAM PROBLEM-SOLVING SEMINAR WEEK 4: FUNCTIONAL EQUATIONS

MARK LUCIANOVIC AND RAVI VAKIL

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

1. Polynomials!

(a) Find all polynomials $f(x)$ such that $f(x) = f(x-1)f(x-2)f(x-3)$.
(b) What polynomials $p(t)$ satisfy

$$p(x+y) + p(y+z) + p(z+x) = p(x) + p(y) + p(z) + p(x+y+z)?$$

2. Solve the following functional equations for *continuous* real-valued functions on $\mathbb{R}$:

(a) $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.
(b) $g(2x) - g(x) = x$ for all $x \in \mathbb{R}$, and $g(0) = 1$.

3 (1992A1). Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:

(i) $f(f(n)) = n$, for all integers n;
(ii) $f(f(n+2)+2) = n$ for all integers n;
(iii) $f(0) = 1$.

4 (1987A3). For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.
(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

Date: Monday, October 31, 2005.

5 (1999A1). Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that, for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

6 (1971B2). Let $F(x)$ be a real valued function defined for all real x except for $x = 0$ and $x = 1$ and satisfying the functional equation

$$F(x) + F((x-1)/x) = 1 + x.$$

Find all functions $F(x)$ satisfying these conditions.

7 (1968 IMO). Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation:

$$f(x + a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2}$$

for some fixed $a > 0$. Prove that f is periodic, and give an example of a non-constant f satisfying the equation for $a = 1$.

8. Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(1) = 1$ and for all n we have

(a) $3f(n)f(2n+1) = f(2n)(1 + 3f(n))$,
(b) $f(2n) < 6f(n)$.

Find all pairs (a, b) such that $f(a) + f(b) = 293$. (Sam Vandervelde)

9 (1991B2). Suppose f and g are nonconstant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

10. Iterations on a theme:

(a) Prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ which satisfies the functional equation $f(f(n)) = n + 1987$. (*Hint*: anything special about the number 1987, besides that it was the year this was an IMO problem?)
(b) Is there an $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(f(n)) = n^2$?
(c) Is there a $g : \mathbb{R} \to \mathbb{R}$ such that $g(g(x)) = -x$? Is there a *continuous* g?

11 (1996A6). Let $c \geq 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f(x^2 + c)$ for all $x \in \mathbb{R}$.

This handout can be found at **http://math.stanford.edu/~vakil/putnam05/**

E-mail address: `markl@math.stanford.edu, vakil@math.stanford.edu`