PUTNAM PROBLEM-SOLVING SEMINAR WEEK 3: COUNTING AND COMBINATORICS

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Things to remember. $n! = 1 \times 2 \times \cdots \times n$. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$.

A. Show that $\binom{r}{0}\binom{s}{n} + \binom{r}{1}\binom{s}{n-1} + \cdots + \binom{r}{n}\binom{s}{0} = \binom{r+s}{n}$.

B. Show that the number of n-tuples of non-negative integers that add to k is $\binom{n+k-1}{n-1}$ (Alok Aggarwal). For experts: show that this problem is equivalent to finding the coefficient of x^k in $(1 + x + x^2 + \cdots)^n = (1-x)^{-n}$.

1. Verify algebraically that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

2. Show that $\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}$.

3. (a) Find all $n \geq 0$ such that $\binom{2n}{n}$ is odd.
(b) What happens if you replace "non-negative" with "positive" in problem B?

4. Show that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n} = 0$ for $n > 0$.

5. (a) Show that the product of m consecutive integers is always divisible by $m!$.
(b) Prove that $\binom{2k}{k} = \frac{2}{\pi}\int_0^{\pi/2}(2\sin\theta)^{2k}d\theta$. (from Razvan Gelca and Titu Andreescu's forthcoming book "Putnam and beyond")
(c) Show that $\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$. (old Putnam problem — and very useful fact)

6. Sum the following.
(a)
$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$$
(b)
$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n}$$

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7. Let p_n denote the nth prime, and let π_n the count of primes less than n. For example:

$$\begin{array}{ll} \{p\}: & 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots \\ \{\pi\}: & 0, 0, 1, 2, 2, 3, 3, 4, 4, 4, \ldots \end{array}$$

Let q_n denote the number of terms of π less than n. What can you say about q_n? (Try a few small cases!) Why is this true? (from this year's poster; I was recently reminded of this problem by Jim Tanton, author of *Solve This!*)

8. Find a combinatorial explanation for the algebraic identity

$$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}.$$

Hint: This is part of the background of the Stanford Math Circle website. (Sam Vandervelde)

9. Sum

$$\sum_{j=0}^{n} \sum_{i=j}^{n} \binom{n}{i}\binom{i}{j}.$$

(*Hint 1:* try small cases. *Hint 2:* "reverse the order of summation". Sketch the i and j that are in the sum to see how to turn this into something like $\sum_{i=?}^{?} \sum_{j=?}^{?} ??$.)

10. If k and m are positive integers, show that the polynomial

$$(x^{k+m} - 1)(x^{k+m-1} - 1) \cdots (x^{k+1} - 1)$$

is divisible by

$$(x^m - 1)(x^{m-1} - 1) \cdots (x - 1).$$

(From Gelca and Andreescu. This is a deformed version of the classical binomial coefficients that appear in computational aspects of quantum physics.)

11. Let S be a set of points in the plane and let R be a fixed positive real number. We want to place discs of radius R on the plane such that,

(i) Every point in S is covered by exactly one disc, and
(ii) Ever disc is centered at a point in S.

This may or may not be possible, but if it is possible, show that all solutions use the exact same number of discs. (David Arthur)

12. Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

(Putnam 1991B4)

This handout can be found at **http://math.stanford.edu/˜vakil/putnam05/**

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