PUTNAM PROBLEM-SOLVING SEMINAR WEEK 2: NUMBER THEORY

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Number theory notions to know: modular arithmetic, unique factorization, greatest common divisor, division algorithm, r, s relatively prime means that there are a and b such that $ar + bs = 1$, Chinese Remainder Theorem, positional notation, Wilson's Theorem $(p-1)! \equiv -1 \pmod p$, Fermat's little theorem $a^p \equiv a \pmod p$, Euler ϕ-function $\phi(n) = \#$ numbers between 1 and n relatively prime to n, $a^{\phi(n)} \equiv 1 \pmod n$ if $\gcd(a, n) = 1$.

1. Is a natural number uniquely determined by the product of its (positive) divisors? (Mark Lucianovic)

2. If $2n+1$ and $3n+1$ are both perfect squares, show that n is divisible by 40.

3. (a) How many zeros does 1000! end with? (b) Is $\binom{100}{36}$ even or odd? (Follow-up question: for how many k is $\binom{100}{k}$ odd? The answer is surprising...)

4. Prove that $\frac{a+b}{c+d}$ is irreducible if $ad - bc = 1$.

5. (a) Show that there are an infinite number of primes of the form $6n-1$. (Hint: if there are only a finite number $p_1, \dots, p_k$, consider $(p_1 \cdots p_k)^2 - 1$.)
(b) Prove that there are an infinite number of primes of the form $4n-1$.

6. Let n be a positive integer. Suppose that 2^n and 5^n *begin* with the same digit. Then there is only one possible value for this common initial digit. Find, with proof, that digit. (Sam Vandervelde, from Engel's book *Problem-Solving Strategies*)

7. What are the last two digits of 3^{1234}?

8. An (ordered) triple (x_1, x_2, x_3) of positive irrational numbers with $x_1 + x_2 + x_3 = 1$ is called *balanced* is $x_i < 1/2$. If a triple is not balanced, say if $x_j > 1/2$, one performs the

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following "balancing act":

$$B(x_1, x_2, x_3) = (x_1', x_2', x_3'),$$

where $x_i' = 2x_i$ if $i \neq j$ and $x_j' = 2x_j - 1$. If the new triple is not balanced, one performs the balancing act on it. Does continuation of this process always lead to a balanced triple after a finite number of performances of the balancing act? (Putnam 1977)

9. Consider two lists. List A consists of the positive powers of 10 $(10, 100, 1000, \dots)$ written in base 2. List B consists of the positive powers of 10 written in base 5. Show that, for any integer $n > 1$, there is exactly one number in exactly one of the lists that is exactly n digits long.

Powers of 10	List A	List B
10	1010 (4 digits)	20 (2 digits)
100	1100100 (7 digits)	400 (3 digits)
1000	1111101000 (10 digits)	13000 (5 digits)
10000	10011100010000 (14 digits)	310000 (6 digits)

(1994 Asian Pacific Mathematical Olympiad)

10. For each positive integer n, write the sum $\sum_{m=1}^{n} \frac{1}{m}$ in the form $\frac{p_n}{q_n}$, where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n. (Putnam 1997 B3)

This handout can be found at **http://math.stanford.edu/~vakil/putnam05/**

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