PUTNAM PROBLEM-SOLVING SEMINAR WEEK 1: INDUCTION AND PIGEONHOLE

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The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

THE PROBLEMS:

1. On a certain street there are twenty houses, ten along each side of the road. Abe, Bill, Cathy, Dierdre, and Ed each live in one of the houses. Prove that among these five people, there must be two of them who live on the same side of the street separated by no more than three houses between them.

2. Choose any eight positive integers from 1 to 100. Prove that among your eight numbers, it is possible to choose two of them whose ratio lies between 1 and 2. (Variation – pick fifty-one numbers instead and show that one of them is a multiple of another.)

3. Consider the sequence defined by $a_1 = 1$ and $a_n = \sqrt{2a_{n-1}}$. Prove that $a_n < 2$ for all n.

4. (Monks on an Island) There is an island inhabited by monks who are very intelligent but have some very rigid rules that they live by. For one, they believe that blue eyes are evil and if a monk finds out he has blue eyes he must kill himself *that very day*. Another rule is that, despite everyone seeing each other every day, they cannot communicate in any way whatsoever. That, combined with the fact that there are no reflective surfaces on the island, means that the blue-eyed monks never find out their eye color and so all is good for years and years.

But one day a visitor comes to the island and before he leaves, remarks for all to hear that he's never seen such beautiful blue eyes as on this island.

Assumptions: - every monk sees every other monk each evening at communal dinner. - it is common knowledge on the island that all monks are rational and obey all monk protocols perfectly. - monks reason instantly and if one concludes that he has blue eyes

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he will commit suicide at midnight of that day. - the visitor's announcement is made publicly at the dinner. - the visitor adds nothing but that observation (that there exist blue eyes on the island).

What happens to the monks?

5. (submitted by Belgian P.-O. Dehaye) How many breaks does it take to divide an $m \times n$ chocolate bar into 1×1 chunks?

6. (Putnam 2002, A2) Prove that given any five points on a sphere, there exists a closed hemisphere which contains four of the points.

7. Prove that in a room with n people, at least two people know exactly the same number of people. Assume knowing is a mutual relation: If Paul knows Pete, then Pete knows Paul.

8. (1991 USAMO #2) For each non-empty subset of $\{1, 2, ..., n\}$ take the sum of the elements divided by the product. Show that the sum of the resulting quantities is $n^2 + 2n - (n+1)s_n$, where $s_n = 1 + 1/2 + 1/3 + ... + 1/n$.

9. (1989 USAMO #2) In a tournament between 20 players, there are 14 games (each between two players). Each player is in at least one game. Show that we can find 6 games involving 12 different players.

10. (Putnam 1997, A3) Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots\right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right) dx.$$

11. Let N be a positive integer. Prove that some multiple of N (written in base ten, of course) consists entirely of 0's and 1's. (HINT: consider differences between repunits—numbers of the form 1111...1.)

12. Prove that a graph with v vertices and no "tetrahedra" (i.e. no complete subgraphs on four vertices) can have at most $\lfloor v^2/3 \rfloor$ edges.

13. The first 2n natural numbers are arbitrarily divided into two groups of n numbers each. The numbers in the first group are sorted in ascending order, i.e., $a_1 < ... < a_n$, and the numbers in the second group are sorted in descending order: $b_1 > ... > b_n$. Find, with proof, the sum $|a_1 - b_1| + ... + |a_n - b_n|$.

14. Consider a simple graph where any pair of vertices is connected by at most one edge and no edge connects a vertex to itself. Each vertex is coloured black or white. Switching a vertex means changing its colour as well that of its immediate neighbours (vertices connected by one edge). Assume that all vertices are white to start with. Prove that we can always perform a series of switches to change all of them to black.

This handout can soon be found at **http://math.stanford.edu/~vakil/putnam05/**

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