# PROBLEM-SOLVING MASTERCLASS WEEK 7

We'll end with an "all-star" session of seven problems.

**1.** Show that

$$\pi \cot(\pi x) = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{x+n}.$$

(Theo Johnson-Freyd, from *Proofs from the Book*, which is an amazing book)

**2.** You have coins $C_1, C_2, \dots, C_n$. For each $k$, $C_k$ is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the $n$ coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of $n$. (Cihan Baran, Putnam 2001A2)

**3.** Let $G_n$ be the geometric mean of the binomial co-efficients in the $n$th row of Pascal's triangle $\binom{n}{0}, \dots, \binom{n}{n}$. Find $\lim_{n \to \infty} G_n^{1/n}$. (Bob Hough, from Andreescu and Galcea's book *Putnam and Beyond*)

**4.** If $p$ is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by $p^2$. (Nathan Pflueger, Putnam 1996A5)

**5.** For positive $a$, $b$, and $c$, show that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a+b+c)^3.$$

(Kiyoto Tamura, likely from a recent USAMO)

**6.** The sequence $u_n$ is defined by $u_0 = 1$, $u_{2n} = u_n + u_{n-1}$, $u_{2n+1} = u_n$. Show that for any positive rational $k$ we can find $n$ such that $\frac{u_n}{u_{n+1}} = k$. (Kiat Chuan Tan, Putnam 2002A5)

**7.** *Pick's theorem!* The area of any (not necessarily convex) polygon $Q \subset \mathbb{R}^2$ with integral vertices is given by:

$$A(Q) = n_{int} + \frac{1}{2} n_{bd} - 1,$$

where $n_{int}$ is the number of integral points in the interior of $Q$ and $n_{bd}$ is the number of integral points on the boundary of $Q$. (Woodley Packard)

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