PROBLEM-SOLVING MASTERCLASS WEEK 6

1. Let p be an odd prime and let $\mathbb{Z}_p$ denote (the field of) integers modulo p. How many elements are in the set

$$\{x^2 : x \in \mathbb{Z}_p\} \cap \{y^2 + 1 : y \in \mathbb{Z}_p\}?$$

(Kiat Chuan Tan, 1991B5)

2. Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \dots, a_n)$ such that $1/a_1 + 1/a_2 + \cdots + 1/a_n = 1$. Determine whether N_{10} is even or odd. (Nathan Pflueger, 1997A5)

3. Suppose f and g are two increasing functions on $\mathbb{R}$. Prove that for any real numbers a and b the inequality

$$(b-a)\int_a^b f(x)g(x)\,dx \geq \int_a^b f(x)\,dx \times \int_a^b g(x)\,dx.$$

(John Hegeman, from Andreescu and Gelca's forthcoming book *Putnam and beyond*)

4. Prove the "logarithmic mean" inequality for $a > b > 0$:

$$\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2}.$$

(Ravi Vakil; # 10 from last week, proposed by Mark Lucianovic)

5. For any integer a, set

$$n_a = 101a - 100 \cdot 2^a.$$

Show that for $0 \leq a, b, c, d \leq 99$, $n_a + n_b \equiv n_c + n_d \pmod{10100}$ implies $\{a, b\} = \{c, d\}$. (Kiat Chuan Tan, 1994B6)

6. Let S be a nonempty closed bounded convex set in the plane. Let K be a line and t a positive number. Let L_1 and L_2 be support lines for S parallel to K, and let $\overline{L}$ be the line parallel to K and midway between L_1 and L_2. Let $B_S(K, t)$ be the band of points whose distance from $\overline{L}$ is at most $(t/2)w$, where w is the distance between L_1 and L_2. What is the smallest t such that

$$S \cap \bigcap_K B_S(K, t) \neq \emptyset$$

for all S? (K runs over all lines in the plane.) (Ravi Vakil, 1990B6)

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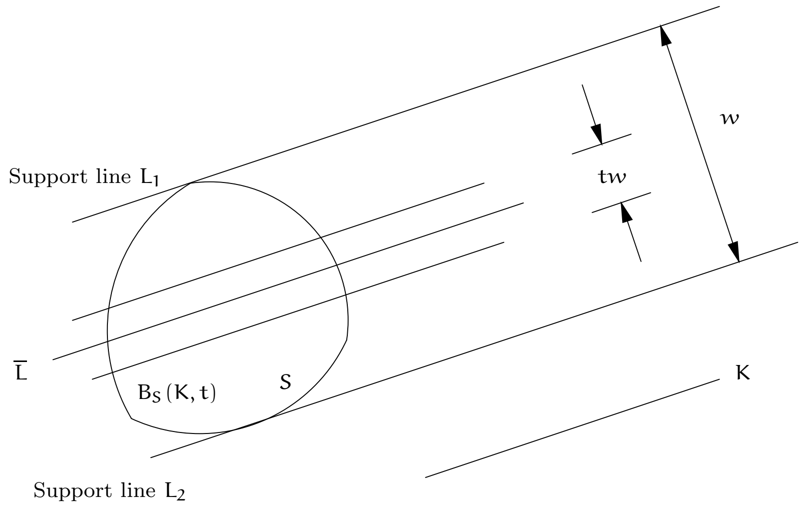
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$B_S(K,t)$
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