PROBLEM-SOLVING MASTERCLASS WEEK 5

1. Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

(Cihan Baran, Putnam 1991B4, # 12 on the combinatorics week problem list)

2. Let a, b, c be positive real numbers with $abc = 1$. Prove that:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq 3/2.$$

(Soren Galatius, IMO1995 # 2)

3. Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 1111 \cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$. (Kiat Chuan Tan, Putnam 1998B5)

4. The hands of an accurate clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly. (Ravi Vakil, Putnam 1983A2)

5. Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2+n+1)$, where $\operatorname{Arccot} t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot\theta = t$. (Marcello Herreshoff, Putnam 1986A3)

6. A sequence of convex polygons $\{P_n\}$, $n \geq 0$, is defined inductively as follows. P_0 is an equilateral triangle with sides of length 1. Once P_n has been determined, its sides are trisected; the vertices of P_{n+1} are the *interior* trisection points of the sides of P_n. Thus P_{n+1} is obtained by cutting corners off P_n, and P_n has $3 \cdot 2^n$ sides. (P_1 is a regular hexagon with sides of length 1/3.) Express $\lim_{n\to\infty} \operatorname{Area}(P_n)$ in the form $\sqrt{a}/b$, where a and b are positive integers. (Ravi Vakil, Putnam 1984B6) (Fun facts from the 1965–84 Putnam book: de Rham shows that the limiting curve is C^1 with zero curvature almost everywhere, but every subarc contains points where the curvature is infinite. Consequently, the curve is nowhere analytic. If the construction is repeated, but with each edge divided in the ratio $(1/4, 1/2, 1/4)$ rather than $(1/3, 1/3, 1/3)$, then the resulting limit curve *is* analytic, consisting of piecewise parabolic arcs!)

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