PROBLEM-SOLVING MASTERCLASS WEEK π (SOMEWHERE BETWEEN 3 AND 4)

1. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3 - x}\right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$. (Kiat Chuan Tan, 1992B4)

2. Let $A(n)$ denote the number of sums of positive integers $a_1 + a_2 + \cdots + a_r$ which add up to n with $a_1 > a_2 + a_3$, $a_2 > a_3 + a_4$, $\dots$, $a_{r-2} > a_{r-1} + a_r$, $a_{r-1} > a_r$. Let $B(n)$ denote the number of $b_1 + b_2 + \cdots + b_s$ which add up to n, with

(i) $b_1 \geq b_2 \geq \cdots \geq b_s$,
(ii) each b_i is in the sequence $1, 2, 4, \dots, g_j, \dots$ defined by $g_1 = 1$, $g_2 = 2$, and $g_j = g_{j-1} + g_{j-2} + 1$, and
(iii) if $b_1 = g_k$ then every element in $\{1, 2, 4, \dots, g_k\}$ appears at least once as a b_i.

Prove that $A(n) = B(n)$ for each $n \geq 1$.

For example, $A(7) = 5$ because the relevant sums are 7, $6+1$, $5+2$, $4+3$, $4+2+1$, and $B(7) = 5$ because the relevant sums are $4+2+1$, $2+2+2+1$, $2+2+1+1+1$, $2+1+1+1+1+1$, $1+1+1+1+1+1+1$. (John Hegeman, Putnam 1991A6)

3. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

(Nathan Pflueger, Putnam 1996B2)

4. Suppose $x_1, \dots, x_n$ are each ± 1, and

$$\sum_{i=1}^{n} x_i x_{i+1} x_{i+2} x_{i+3} = 0,$$

where x_{n+i} is interpreted as x_i. Show that n is divisible by 4. (Ravi Vakil)

5. Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.) (Kiat Chuan Tan, 1989A3)

6. Show that there is some N so that if you take any N points on the plane, no three on a line, then you can find 2005 of them that form a convex 2005-gon. (Ravi Vakil, from a

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book by Polya, Tarjan, and Woods — Polya was professor here long ago, famous for his books on problem-solving)

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