PROBLEM-SOLVING MASTERCLASS WEEK 2

1. If $2n + 1$ and $3n + 1$ are both perfect squares, show that n is divisible by 40. (Steph Abegg, # 2 from last week's seminar; Cihan Biran will give a second proof)

2. If $A_1 + \cdots + A_n = \pi$, $0 < A_i < \pi$, $i = 1, \ldots, n$, then show that

$$\sin A_1 + \cdots + \sin A_n \leq n \sin \pi/n.$$

(Brian Munson)

3. Let $A(n)$ denote the number of sums of positive integers $a_1 + a_2 + \cdots + a_r$ which add up to n with $a_1 > a_2 + a_3$, $a_2 > a_3 + a_4$, ..., $a_{r-2} > a_{r-1} + a_r$, $a_{r-1} > a_r$. Let $B(n)$ denote the number of $b_1 + b_2 + \cdots + b_s$ which add up to n, with

(i) $b_1 \geq b_2 \geq \cdots \geq b_s$,
(ii) each b_i is in the sequence $1, 2, 4, \ldots, g_j, \ldots$ defined by $g_1 = 1$, $g_2 = 2$, and $g_j = g_{j-1} + g_{j-2} + 1$, and
(iii) if $b_1 = g_k$ then every element in $\{1, 2, 4, \ldots, g_k\}$ appears at least once as a b_i.

Prove that $A(n) = B(n)$ for each $n \geq 1$.

For example, $A(7) = 5$ because the relevant sums are 7, $6 + 1$, $5 + 2$, $4 + 3$, $4 + 2 + 1$, and $B(7) = 5$ because the relevant sums are $4 + 2 + 1$, $2 + 2 + 2 + 1$, $2 + 2 + 1 + 1 + 1$, $2 + 1 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1 + 1 + 1$. (John Hegeman, Putnam 1991A6 — we might do this one next week)

4. Suppose $x_1, \ldots, x_n$ are each ± 1, and

$$\sum_{i=1}^{n} x_i x_{i+1} x_{i+2} x_{i+3} = 0,$$

where x_{n+i} is interpreted as x_i. Show that n is divisible by 4. (Ravi Vakil)

5. $2n$ points are drawn on the circumference of a circle. In how many ways can these points be joined in pairs by n chords which do not intersect within the circle? (Alok Aggarwal)

6. Show that there is some N so that if you take any N points on the plane, no three on a line, then you can find 2005 of them that form a convex 2005-gon. (Ravi Vakil, from a book by Polya, Tarjan, and Woods — Polya was professor here long ago, famous for his books on problem-solving)

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