PROBLEM-SOLVING MASTERCLASS WEEK 1

1. Consider two lists. List A consists of the positive powers of 10 (10, 100, 1000, $\ldots$) written in base 2. List B consists of the positive powers of 10 written in base 5. Show that, for any integer $n > 1$, there is exactly one number in exactly one of the lists that is exactly n digits long.

Powers of 10	List A	List B
10	1010 (4 digits)	20 (2 digits)
100	1100100 (7 digits)	400 (3 digits)
1000	1111101000 (10 digits)	13000 (5 digits)
10000	10011100010000 (14 digits)	310000 (6 digits)

(Ravi Vakil, a problem I made up long ago, that appeared on the 1994 Asian Pacific Mathematical Olympiad)

2. Let $f(x)$ be differentiable on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. For each positive integer n and arbitrary given positive numbers $k_1, k_2, \ldots, k_n$, show that there exist distinct x_1, x_2, $\ldots$, x_n such that

$$\frac{k_1}{f'(x_1)} + \frac{k_2}{f'(x_2)} + \cdots + \frac{k_n}{f'(x_n)} = k_1 + k_2 + \cdots + k_n.$$

(Bob Hough, from Larson 6.6.9)

3. For a positive real number r, let $G(r)$ be the minimum value of $\left|r - \sqrt{m^2 + 2n^2}\right|$ for all integers m and n. Prove or disprove the assertion that $\lim_{r\to\infty} G(r)$ exists and equals 0. (Theo Johnson-Freyd, Putnam 1986 B4)

4. The first $2n$ natural numbers are arbitrarily divided into two groups of n numbers each. The numbers in the first group are sorted in ascending order, i.e., $a_1 < \cdots < a_n$, and the numbers in the second group are sorted in descending order: $b_1 > \cdots > b_n$. Find, with proof, the sum

$$|a_1 - b_1| + \cdots + |a_n - b_n|.$$

(Paul-Olivier Dehaye)

5. Consider a regular n-gon inscribed in a unit circle with vertices labeled (cyclically) P_1, $\ldots$, P_n. Show that

$$|P_1P_2|\,|P_1P_3|\cdots|P_1P_n| = n.$$

(Ravi Vakil)

Date: Monday, October 17, 2005.