PUTNAM PROBLEM SOLVING SEMINAR WEEK 2: LINEARLY RECURRENT SEQUENCES

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

The problems.

1.

(a) The sequence q_1, q_2, ... satisfies $q_n = 3q_{n-2} - 2q_{n-3}$, and $q_0 = 0$, $q_1 = 3$, $q_2 = 11$. Find a general formula for q_n.

(b) What is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$?

(c) The sequence r_1, r_2, ... satisfies $r_n = (5/2)r_{n-1} - r_{n-2}$, and $r_1 = 2003$. Suppose the sequence converges to a finite real number. Find r_2.

(d) The sequence G_0, G_1, G_2, ... consists of every other Fibonacci number. Show that there is a linear recursion (e.g. of the form $G_n = aG_{n-1} + bG_{n-2}$). (Follow-up: How about a sequence consisting of every *tenth* Fibonacci number. How do you know there's a recursion? Harder: With integer coefficients?)

(e) Use the theory of linear recursive sequences to find a formula for the sequence $s_0 = 1$, $s_1 = 2$, $s_n = s_{n-2}$. What do you observe? Now try a sequence with period four, such as $t_0 = 1$, $t_1 = 0$, $t_2 = 0$, $t_3 = 0$.

(f) Find a recurrence satisfied by $f_n = 3^n + 4^{n+1}$.

(g) Find a recurrence satisfied by all cubic polynomials.

(h) Suppose $f_2 = 2$, and $f_n = -2f_{n-1} - 4f_{n-2}$. Find f_{2003}. (It looks like there isn't enough information to solve this problem.)

(i) Find a length two recurrence satisfied by $C_n = \cos n^\circ$.

2. Suppose f_n is a sequence of rational numbers, with f_0 and f_1 not both zero, such that $f_n = f_{n-1} + f_{n-2}$. Show that f_n is unbounded as $n \to \infty$. Is this still true if the condition of rationality is removed?

3. (a) Find all sequences satisfying $f_{n+2} = 5f_{n+1} + 6f_n + 2^n$. (b) Find a closed-form expression for the sequence satisfying $f_n = f_{n-1} + n2^n$, $f_0 = 0$. (You can translate this as:

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find

$$1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n,$$

but I would rather that you solve it by considering it as a recurrence relation.)

4. (a) An elf skips up a flight of numbered stairs, starting at step 1 and going up one or two steps with each leap. He counts how many ways he can reach the nth step, calling the nth number E_n. What is E_n? (b) How many n-digit numbers are there, each of whose digits is either 0 or 1, with no 1's adjacent?

5. Alice and Bob play the following game. They flip a coin until the first "heads" comes up. If this happens on an odd flip, Alice wins; if it happens on an even flip, Bob wins.

(a) What is the probability that the game is over by the end of the nth flip? What is the probability that Alice has won by this time? What is the probability that Bob has won by this time?
(b) Hence show that the game will end in finite time with probability 1, and find Alice's odds of winning.

6. A gambling graduate student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring exactly n points at some time in a sequence of n tosses is $(2 + (-1/2)^n)/3$. (Hint: Let P_n denote the probability of scoring exactly n points at some time. Express P_n in terms of P_{n-1}, or in terms of P_{n-1} and P_{n-2}. Use this linear recursion to give an inductive proof. Even better hint, useful in many circumstances: you've been given the answer, so reverse-engineer the recursion, and then try to prove it.)

7. (This isn't a *linear* recurrence question, but it *is* a neat recurrence question.)

(a) Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$. Find a recurrence relation for I_n.
(b) Show that

$$I_{2n} = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n)} \cdot \frac{\pi}{2}.$$

(c) Show that

$$I_{2n+1} = \frac{2 \times 4 \times 6 \times \cdots \times (2n-2)}{1 \times 3 \times 5 \times \cdots \times (2n-1)}.$$

(Fun follow-up: Write these formulas in terms of factorials. Hint: Can you see why $1 \times 3 \times \cdots \times (2n-1) = (2n)!/(2^n n!)$? Then try plugging $n = 1/2$ into the formula you get for (b); what do you get for $(1/2)!$? What's that $\sqrt{\pi}$ doing there?!)

8. Solve the double recurrence

$$\begin{aligned} f_n &= f_{n-1} - 3g_{n-1} \\ g_n &= -3f_{n-1} + 9g_{n-1} \end{aligned}$$

(One possible approach: find a recurrence involving just f or g.) If you solve this and don't use eigenvalues and eigenvectors (you don't need them!), please tell me, and I'll teach you about them.

9. f is a strictly increasing real-valued function on the reals. It has inverse f^{-1}. Find all possible f such that $f(x)+f^{-1}(x)=2x$ for all x. (Hint: how is this a recurrence problem?) (APMO 1989)

10. A triangle has sides a, b, and c. Construct another triangle with sides $(-a+b+c)/2$, $(a-b+c)/2$, $(a+b-c)/2$ (if possible). For which triangles an the process be repeated arbitrarily many times? (APMO 1992)

11. Let $T_0=2$, $T_1=3$, $T_2=6$, and for $n\geq 3$,

$$T_n=(n+4)T_{n-1}-4nT_{n-2}+(4n-8)T_{n-3}.$$

The first few terms are

$$2,3,6,14,40,152,784,5168,40576,363392.$$

Find, with proof, a formula for T_n of the form $T_n=A_n+B_n$, where (A_n) and (B_n) are well-known sequences. (Hint: use all tools at your disposal, including inspired guesswork. Play around with the numbers. What do you notice?) (Putnam 1990)

12. *(Ordinary differential equations with constant coefficients.)* Solve the differential equation $f''(x)=5f'(x)-6f(x)$, with initial conditions $f(0)=0$, $f'(0)=1$. (Translation: develop the general theory of such equations.) *Hint:* use the characteristic equation to guess two solutions. To guess a solution, examine a simpler equation of the same type, such as $g'(x)=7g(x)$, and find its solutions. (What would happen if the differential equation had a repeated root? Try it with $f''(x)=2f'(x)-f(x)$.

13. A population of rabbits and a population of hedgehogs are competing to survive in a wood. Suppose that their populations at time x are given by $f(x)$ and $g(x)$, and that they evolve according to the differential equations

$$\begin{aligned} f'(x) &= \frac{1}{16}f(x)-\frac{3}{16}g(x) \\ g'(x) &= -\frac{3}{16}f(x)+\frac{9}{16}g(x). \end{aligned}$$

(We won't worry about the fact that rabbits and hedgehogs tend to come in integer quantities. If $f(x)$ is 0 at some time x, however, then it stays 0, and similarly for $g(x)$.)

(a) They approach an equilibrium. What is it? Is it stable?

(b) With arbitrary initial conditions, will they always approach an equilibrium?

(c) (logically independent of the previous two) Find a constant a such that $f(x)+ag(x)$ is constant.

Sample recurrence write-up.

Problem. Solve the linearly recurrent equation $f_n=5f_{n-1}-6f_{n-2}$, $f_0=0$, $f_1=1$.

Solution. The characteristic equation for this recurrence is $t^2-5t+6=0$, which has solutions $t=2$ and $t=3$ (each with multiplicity 1. Thus the solutions to the recurrence are all of the form $A2^n + B3^n$ (for constants A and B). Using the values at $n = 0$ and $n = 1$, we find that $A = -1$ and $B = 1$. Hence the solution is $f_n = 3^n - 2^n$.

This handout can (soon) be found at

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