# PUTNAM PROBLEM SOLVING SEMINAR WEEK 1: INDUCTION, PIGEONHOLE, AND MISCELLANEOUS PROBLEM-SOLVING

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**The Rules.** These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on. If you would like to practice with the Pigeonhole Principle or Induction (a good idea if you haven't seen these ideas before), try those problems.

**The Hints.** Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

**The problems.**

*The first problems are for practice with the pigeonhole principle.*

**1.** Prove that there are two people in the U.S. right now with the same amount of hair on their heads (not including bald people!).

**2.** Let $A$ be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, ..., 100. Prove that there must be two distinct integers in $A$ whose sum is 104.

**3.** Show that if there are $n$ people at a party, then two of them know the same number of people (among those present).

**4.** Five points lie in an equilateral triangle of size 1. Show that two of the points lie no farther than $1/2$ apart. Can the "$1/2$" be replaced by anything smaller? Can it be improved if the "five" is replaced by "six"?

**5.** A lattice point in the plane is a point $(x, y)$ such that both $x$ and $y$ are integers. Find the smallest number $n$ such that given $n$ lattice points in the plane, there exist two whose midpoint is also a lattice point.

**6.** Let $u$ be an irrational real number. Let $S$ be the set of all real numbers of the form $a + bu$, where $a$ and $b$ are integers. Show that $S$ is dense in the real numbers, i.e. for any

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real number $x$ and any $\epsilon > 0$, there is an element $y \in S$ such that $|x - y| < \epsilon$. (Hint: first let $x = 0$.)

**7.** For every $n$ in the set $\mathbb{Z}^+ = \{1, 2, \ldots,\}$ of positive integers, let $r_n$ be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers $c$ and $d$ with $c + d = n$. Find, with proof, the smallest positive real number $g$ with $r_n \leq g$ for all $n \in \mathbb{Z}^+$.

**8.** Prove that there is some integral power of 2 that begins $2002$....

**9.** Given any $n + 1$ integers between $1$ and $2n$, show that one of them is divisible by another.

**10.** Prove that in any group of six people there are either three mutual friends or three mutual strangers. (Hint: Represent the people by the vertices of a regular hexagon. Connect two vertices with a red line segment if the couple represented by these vertices are friends; otherwise, connect them with a blue line segment. Consider one of the vertices, say $A$. At least three line segments emanating from $A$ have the same color. There are two cases to consider.)

**11.** A polygon in the plane has area 1.2432. Show that it contains two distinct points $(x_1, y_1)$ and $(x_2, y_2)$ that differ by $(a, b)$, where $a$ and $b$ are integers.

*The next few problems are for practice with induction.*

**12.** Show that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

**13.** Show that for all positive integers $n$, $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer.

**14.** Show that $1 + 1/\sqrt{2} + 1/\sqrt{3} + \cdots + 1/\sqrt{n} < 2\sqrt{n}$.

**15.** Show that $\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{100}+\sqrt{99}} = 9$.

**16.** Prove that all even perfect squares are divisible by 4. Prove that all odd perfect squares leave a remainder of 1 upon division by 8. (This is a useful fact to know!) What are the possible remainders when you divide a perfect square by 3?

*The last problems are not on any particular topic, and may be a bit harder.*

**17.** Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area $\geq 5/2$.

**18.** How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating $1$'s and $0$'s, beginning and ending with 1?

**19.** If $\alpha$ is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is $\alpha$? (An honest coin is one for which the probability of heads and the probability of tails are both $1/2$. A game is finite if with probability 1 it must end in a finite number of moves.)

**20.** Prove tht a list can be made of all the subsets of a finite set in such a way that

(i) the empty set is the first in the list,
(ii) each subset occurs exactly once, and
(iii) each subset in the list is obtained either by adding one element to the preceding subset of by deleting one element of the preceding subset.

*This handout can (soon) be found at*

**http://math.stanford.edu/˜vakil/putnam03/**

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