PROBLEM SOLVING MASTERCLASS WEEK 2

1. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

(Alex, Putnam 1998A3)

2. Prove that for $n \geq 2$,

$$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}n \equiv \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}n-1 \pmod{n}.$$

(Yuanli, Putnam 1997B5)

3. Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even. (Youngjun, Putnam 2002A3)

4. The vertices of a triangle are lattice points in the plane. Show that the diameter of its circumcircle does not exceed the product of its side lengths. (Paul, Putnam 1971A3)

5. For a positive real number r, let $G(r)$ be the minimum value of $\left|r - \sqrt{m^2 + 2n^2}\right|$ for all integers m and n. Prove or disprove the assertion that $\lim_{r\to\infty} G(r)$ exists and equals 0. (Frank, Putnam 1986B4)

6. Let A be a matrix in $SO(n)$, i.e. an $n \times n$ matrix with determinant 1, whose n column vectors are orthonormal. If $0 < k < n$, show that the determinant of the $k \times k$ matrix in the upper-left corner equals the determinant of the $(n-k) \times (n-k)$ matrix in the lower-right corner. (One interesting consequence: Show that the area of the shadow of a unit cube is equal to its "height" (the difference in height between its highest and lowest points). Hence find the area of the largest possible shadow of a unit cube, and of the smallest.) (Ravi)

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Date: Monday, November 3, 2003.