PROBLEM SOLVING MASTERCLASS WEEK 1

1. Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries. (Paul, Putnam 1994A4)

2. Find the smallest integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board. (Chee Hau, USAMO 2000)

3. Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m,n). (Frank, Putnam 1985B3)

4. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

(Alex, Putnam 1998A3)

5. Prove that for $n \geq 2$,

$$2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}^{n} \equiv 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}^{n-1} \pmod{n}.$$

(Yuanli, Putnam 1997B5)

6. Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even. (Youngjun, Putnam 2002A3)

7. Let A be a matrix in $SO(n)$, i.e. an $n \times n$ matrix with determinant 1, whose n column vectors are orthonormal. If $0 < k < n$, show that the determinant of the $k \times k$ matrix in the upper-left corner equals the determinant of the $(n-k) \times (n-k)$ matrix in the lower-right corner. (One interesting consequence: Show that the area of the shadow of a unit cube is equal to its "height" (the difference in height between its highest and lowest points). Hence find the area of the largest possible shadow of a unit cube, and of the smallest.) (Ravi)

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