PUTNAM PROBLEM SOLVING SEMINAR
WEEK 7: GRAPH THEORY

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The Rules. There are way too many problems here to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Try to turn the first problems into graph problems (if they don't start that way). Think about degrees of vertices. Generalize. Try small and extreme cases. Look for patterns. Use induction. Eat pizza. Work in groups. Use lots of paper. Talk it over. Choose effective notation. Don't give up after five minutes.

The Problems. *The problems are VERY APPROXIMATELY ordered from "easiest" to "hardest."*

1. At a dinner party people shake hands as they are introduced. Not everyone shakes hands with everyone else (some of them already know each other!). Show that there have to be two people who shake hands the same number of times. Show that the number of people who have shaken hands an odd number of times is even.

2. The *adjacency matrix* of a graph G with n vertices is the $n \times n$ matrix A with $A_{ij} = 1$ if there is an edge joining v_i and v_j, and $A_{ij} = 0$ otherwise. What does A^2 mean? A^k?

3. A graph G is *simple* if it has no loops or multiple edges. It is *connected* if you can walk from any vertex to any other vertex along edges. If $\{v_1, \dots, v_k\}$ is a subset of the vertices of G, the induced subgraph is the graph whose vertices are $\{v_1, \dots, v_k\}$ and whose edges are the edges of G with both endpoints in $\{v_1, \dots, v_k\}$.

(1) Let G be a simple graph with no isolated vertices (vertices of degree 0) and no induced subgraphs with exactly two edges. Show that G is the complete graph.
(2) The graph C_4 is the cycle with four vertices (a square). The graph P_4 is the path with four vertices (a line). Let G be a connected simple graph that does not have P_4 or C_4 as induced subgraphs. Prove that G has a vertex adjacent to all other vertices.

4. In the country of Jetlaggia it is possible to travel by air between any two of the main cities; if there is not a direct flight there is at least an indirect flight passing through other cities on the way. A *path* is an air route between two different cities that passes through no intermediate (or start or end) city more than once. The *length* of a path is the total number of cities on it, counting its endpoint but not its starting point. Let M be the maximum of

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all path lengths in Jetlaggia. Prove that any two paths of length M must have at least one city in common.

5. How many people do we need to have at a party to ensure that there are always at least three people all of whom know each other, or three people none of whom know each other? What if three is changed to four? (Hint below).

6. Prove that for any five points in the plane with no three on a line, there are always four which form a convex quadrilateral without the fifth point in the interior.

This is known as the "Happy Ending Problem". It was first observed by Esther Klein, and was generalized later by George Szekeres and Paul Erdos showing that for every k there is some number N for which if there are more than N points in the plane with no three on a line then there is some convex k-gon. The "Happy Ending" is that Klein and Szekeres later married.

7. A set of 1990 people is divided into non-intersecting subsets in such a way that

(1) no one in a subset knows all the others in the subset;
(2) among any three people in a subset there are always at least two who do not know each other;
(3) for any two people in a subset who do not know each other, there is exactly one person in the same subset knowing both of them.

Prove that within each subset every person has the same number of acquaintances. What is the maximum possible number of subsets? (APMO 1990)

8. *Instant Insanity* is a puzzle consisting of four cubes with faces coloured red, blue, green and yellow. Each cube has at least one face of each colour, and the puzzle is to stack them in a tower so that all four colours appear on all four sides of the tower. The specific colours are in Figure 1 below.

9. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is 1.

10. Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares. (Putnam, 2001). (Hint below).

Hints:

- For question 5 with four the answer is $\frac{1}{2}(2003 - 1967)$ (obscured so you can't look at it by accident).

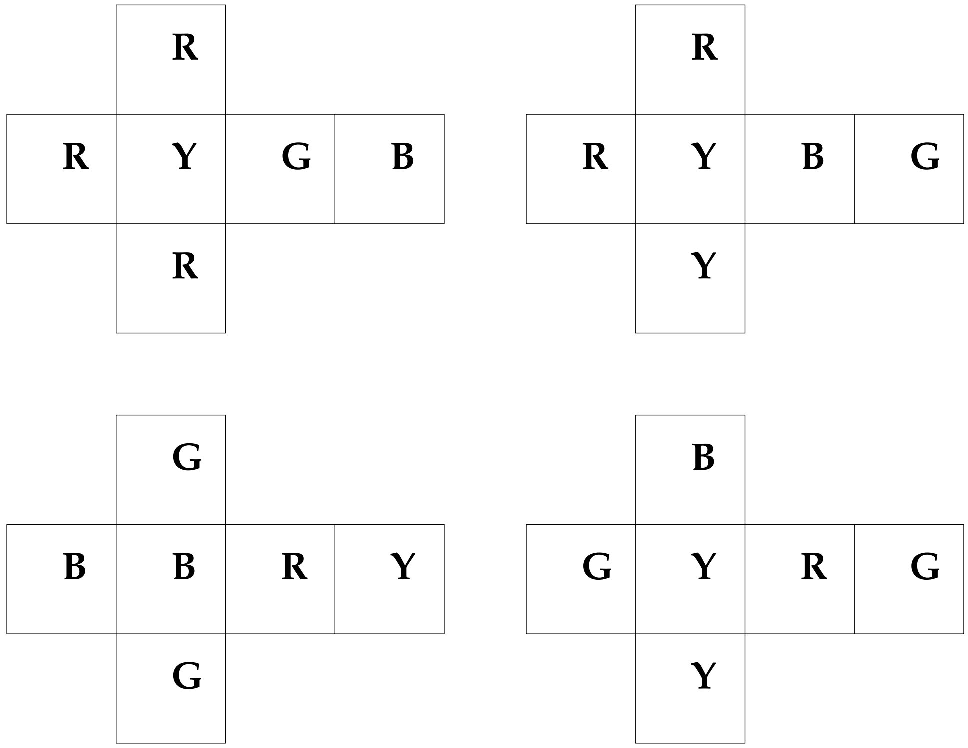


FIGURE 1. Nets for the cubes in the Instant Insanity puzzle

- For question 10, once you've turned it into graph theory, find out what we know about matchings.

A1/A2/B1/B2 Problems from the 1998–2000 Putnams

As a final practice for the Putnam, here are some actual Putnam problems.

1998A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

1998A2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position, of s.

1998B1. Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x>0$.

1998B2. Given a point (a,b) with $0<b<a$, determine the minimum perimeter of a triangle with one vertex at (a,b), one on the x-axis, and one on the line $y=x$. You may assume that a triangle of minimum perimeter exists.

1999A1. Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that, for all x,

$$|f(x)|-|g(x)|+h(x)=\begin{cases}-1 & \text{if } x<-1\\ 3x+2 & \text{if } -1\le x\le 0\\ -2x+2 & \text{if } x>0.\end{cases}$$

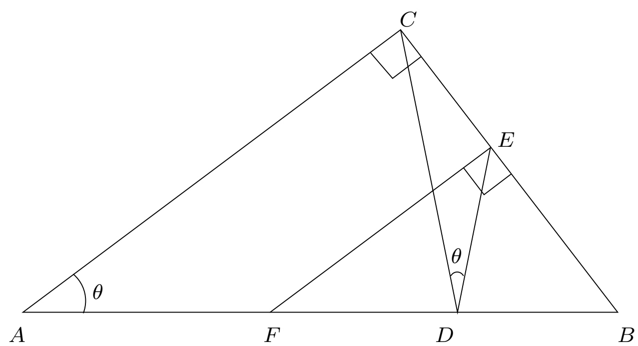
1999A2. Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x),\dots,f_k(x)$ such that

$$p(x)=\sum_{j=1}^{k}(f_j(x))^2.$$

1999B1. Right triangle ABC has right angle at C and $\angle BAC=\theta$; the point D is chosen on AB so that $|AC|=|AD|=1$; the point E is chosen on BC so that $\angle CDE=\theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0}|EF|$. [Here $|PQ|$ denotes the length of the line segment PQ.]

1999B2. Let $P(x)$ be a polynomial of degree n such that $P(x)=Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots. [The roots may be either real or complex.]

2000A1. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty}x_j^2$, given that $x_0,x_1,\dots$ are positive numbers for which $\sum_{j=0}^{\infty}x_j=A$?



2000A2. Prove that there exist infinitely many integers n such that n, $n+1$, $n+2$ are each the sum of two squares of integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

2000B1. Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume, for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

2000B2. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

This handout can be found at

http://math.stanford.edu/∼vakil/putnam03/

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