PUTNAM PROBLEM SOLVING SEMINAR WEEK 6: GEOMETRY AND INTEGRATION

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Useful geometry facts, in a nutshell. Cavalieri's principle. Fubini's theorem (count the same thing in two different ways, or change the order of integration). Approximate polygons with smooth curves and vice versa. Divide shapes into pieces. Don't be afraid to use integration.

The Problems.

1. **(a)** Given a convex polygon S of area A and perimeter p, what is the area of the set of points which lie within distance 1 of S? **(b)** Within distance r? **(c)** What if S is a convex region bounded by a smooth curve?

2. Let S be a shape on a tilted plane in $\mathbb{R}^3$, with area A, and let A_1, A_2, A_3 be the areas of the projections of S on the coordinate planes $\{x_1 = 0\}$, $\{x_2 = 0\}$, $\{x_3 = 0\}$ respectively. Show that $A^2 = A_1^2 + A_2^2 + A_3^2$. Can this be generalised?

3. [Archimedes] Let S be a unit sphere and let C be a cylinder of radius 1 and height 2. Assume the axis of C is vertical, and that the centers of S and C lie at the same height. Let H_1 and H_2 be a pair of horizontal planes intersecting S nontrivially. **(a)** Cover the outsides of S and C by a layer of paint of thickness ϵ. Show that H_1 and H_2 contain between them the same amount of paint from the sphere as from the cylinder. **(b)** Show that the segments of S and C cut out between the two planes have the same surface area.

4. Let S be a sphere embedded inside a cone C, so that they have a circle of tangency. Let H_1 and H_2 be spheres centered on the apex of C, whose surfaces have nonempty intersection with S. **(a)** Show that the segments of S and C cut out between the two spherical surfaces have the same surface area. **(b)** Even without solving part (a), identify three different limiting cases of the theorem. What does the theorem imply in each case?

5. Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown in the Figure 1. Let $A(X)$ denote the area of polygon X. Find the maximum value, or show

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that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

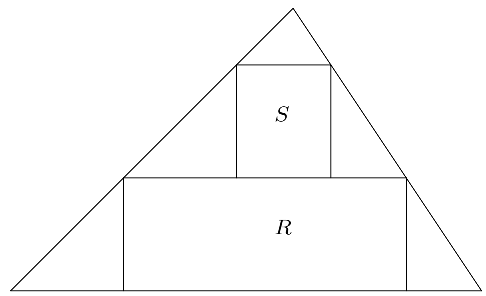

FIGURE 1

6. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b}+c)/d$, where a, b, c, d are positive integers.

7. Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

8. Given a convex polyhedron Q, show that the volume of the set of points which lie within distance r of Q is a cubic polynomial $a + br + cr^2 + dr^3$. Which of a, b, c, d can you interpret? What if Q is a rectangular parallelepiped of side-lengths l, m, n?

9. Let P be a convex polygon in the plane. Project P orthogonally onto a random line through the origin. Show that the expected length of the projection of P is equal to the perimeter of P, up to a constant scale factor. What is the factor?

10. Let Q be a convex polyhedron. **(a)** Show that the expected area of the shadow of Q when projected orthogonally onto a random plane is equal to the surface area (up to a factor). **(b)** Show that the expected area of the length of Q when projected orthogonally onto a random line is equal to the coefficient c in question 8 (up to a factor). First try the special case where Q is rectangular parallelepiped.

11. **(a)** Let Q be a closed polyhedron, not necessarily convex, and define the Euler characteristic $\chi(Q)$ to be $V - E + F$, where V, F, E are the number of vertices, edges and faces of Q. Show that $2\pi\chi(Q)$ is equal to the sum of the angular deficits at the vertices of Q. [The angular deficit at a vertex is defined to be 2π minus the sum of the interior angles at that vertex.] **(b)** When Q is convex, relate the sum of the angle deficits to the construction in question 8, and deduce from this that $\chi(Q) = 2$.

12. A unit cube is positioned in $\mathbb{R}^3$ in some orientation and projected projected onto the coordinate plane $\{x_1 = 0\}$. What is the largest possible area of this projection?

13. Let Q be a cube of side-length 1 foot, suspended, in some orientation, above a flat earth, and illuminated by a sun whose rays are purely vertical. Let A be the area of the shadow of Q in square feet, and let h be the height of Q in feet. Show that $A = h$. Can this be generalised?

14. The 60-inch rule for cabin baggage on aeroplanes allows you to carry on an item of baggage whose "length" is at most 60 inches. All baggage is assumed to be rectangularly parallelepipedal in shape, and "length" is defined to be the sum of the three side lengths. Is this a good definition? Show that you cannot cheat by putting a case inside a bigger case whose length is smaller.

15. Let H be a regular hexagon of side length 7, and suppose that H is tiled with unit diamond-shapes, built by gluing together pairs of unit equilateral triangles. Show that the tiling contains 49 tiles in each of three different orientations.

16. [Milnor's Theorem] Let C be a closed polygonal curve embedded in $\mathbb{R}^3$. Let $\kappa(C)$ denote the sum of the angle-jumps at each vertex. Milnor's theorem asserts that $\kappa(C)$ is greater than 4π if C is knotted. [The angle-jump at a vertex is equal to π minus the angle between the two edges.] **(a)** If C is embedded in $\mathbb{R}^2$, under what conditions is $\kappa(C) = 2\pi$? **(b)** Take the dot product of C with a randomly-chosen unit vector. What is the probability that a given vertex is a local maximum of this function? **(c)** What is the expected number of local maxima when C is dotted with a randomly-chosen unit vector? **(d)** Show that if $\kappa(C) < 4\pi$ then C must be unknotted. **(e)** What is the corresponding theorem when C is a smooth embedded closed curve?

17. Let S be a closed convex subset of $\mathbb{R}^2$ which contains the origin in its interior. For each unit vector $v = [\cos\theta, \sin\theta]$, let $r(\theta) = \max\{r \in \mathbb{R} : rv \in S\}$ and let $u(\theta) = \max\{s \cdot v : s \in S\}$. What is the area of S in terms of the function r? In terms of u? How about the perimeter? Verify that any formulae you obtain are independent of the choice of origin; this is easier with the formulae that depend only on u.

This handout can (soon) be found at

http://math.stanford.edu/~vakil/putnam03/

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