PUTNAM PROBLEM SOLVING SEMINAR
WEEK 5: SUMS AND SERIES

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The Rules. There are way too many problems here to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Look for symmetry. Interchange summation order. Replicate the sum. Telescope. Manipulate a known series. Generalize. Compare with an integral. Try small and extreme cases. Look for patterns. Use induction. Eat pizza. Work in groups. Use lots of paper. Talk it over. Choose effective notation. Don't give up after five minutes.

The Problems. *The problems are APPROXIMATELY ordered from "easiest" to "hardest."*

1. The Riemann zeta function is defined as

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots$$

Show that

$$\text{(i) } \sum_{k=2}^{\infty} (\zeta(k) - 1) = 1. \qquad \text{(ii) } \sum_{k=1}^{\infty} (\zeta(2k) - 1) = \frac{3}{4}.$$

2. Let $\mathcal{S}$ be the set of positive integers whose only prime factors are 2, 3, or 5. Evaluate

$$\sum_{x \in \mathcal{S}} \frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \cdots$$

3. The Fibonacci sequence is defined by $f_0 = f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Evaluate

$$\text{(i) } \sum_{n=1}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}}. \qquad \text{(ii) } \sum_{n=1}^{\infty} \frac{1}{f_{n-1} f_{n+1}}.$$

4. Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n 3^m + m 3^n)}.$$

5. Evaluate

$$\sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{2002^2} + \frac{1}{2003^2}}.$$

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6. Define $\{x_n\}$ by the recurrence $x_1 = 1/2,\ x_{n+1} = x_n^2 + x_n$ for $n \geq 1$. Evaluate

$$\left\lfloor \frac{1}{x_1+1} + \frac{1}{x_2+1} + \cdots + \frac{1}{x_{2003}+1} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

7. Sums of powers. For positive integers k, n define

$$S_k = 1^k + 2^k + \cdots + n^k.$$

Show that

$$\binom{k+1}{1} S_1 + \binom{k+1}{2} S_2 + \cdots + \binom{k+1}{k} S_k = (n+1)^{k+1} - (n+1).$$

Use this recurrence to find a formula for S_2, S_3, and S_4.

8. Prove that the average of the numbers $n \sin n^\circ$, $n = 2, 4, 6, \ldots, 180$, is $\cot 1^\circ$.

9. For $0 < x < 1$, express

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$$

as a rational function of x.

10. Evaluate

$$\text{(i)} \sum_{n=1}^{\infty} \frac{n}{n^4 + n^2 + 1}. \qquad \text{(ii)} \prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

11. For positive integer n, evaluate

$$\sum_{k=0}^{\infty} \left\lfloor \frac{n + 2^k}{2^{k+1}} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

12. For nonnegative integers n and k, define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n, k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k - 2j}.$$

13. Let $f_0(x) = e^x$ and $f_{n+1}(x) = x f_n'(x)$ for $n = 0, 1, 2, \ldots$. Show that

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e.$$

14. Show that the power series for the function

$$e^{ax}\cos(bx), \qquad (a, b > 0),$$

in powers of x has either no zero coefficients, or infinitely many zero coefficients.

15. Define $C(\alpha)$ to be the coefficient of x^{2003} in the power series about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{2003} \frac{1}{y+k} \right) dy.$$

16. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) dx.$$

17. Evaluate

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left(\left\lfloor \frac{2n}{k} \right\rfloor - 2 \left\lfloor \frac{n}{k} \right\rfloor \right),$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

18. Let $B(n)$ be the number of ones in the base two expression for the positive integer n. For example, $B(6) = B(110_2) = 2$ and $B(15) = B(1111_2) = 4$. Determine whether or not

$$\exp\left(\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)} \right)$$

is a rational number.

19. Show that the power series respresentation for the series

$$\sum_{n=0}^{\infty} \frac{x^n (x-1)^{2n}}{n!}$$

cannot have three consecutive zero coefficients.

20. Find

$$\lim_{n \to \infty} \frac{1}{n^5} \sum_{h=1}^{n} \sum_{k=1}^{n} \left(5h^4 - 18h^2k^2 + 5k^4 \right).$$

21. Evaluate

$$\lim_{n \to \infty} \frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n}.$$

Appendix.

General binomial theorem for real α:

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n,$$

where the binomial coefficient is defined for integer $n \geq 0$ by

$$\binom{\alpha}{n} \doteq \frac{(\alpha)(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Useful series:

$$\begin{aligned}
(1+x)^m &= \sum_{n=0}^{m} \binom{m}{n} x^n, \quad \forall x. \\
\frac{1-x^m}{1-x} &= \sum_{n=0}^{m-1} x^n, \quad x \neq 1. \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \quad |x| < 1. \\
\frac{x}{(1-x)^2} &= \sum_{n=0}^{\infty} n x^n, \quad |x| < 1. \\
(1-x)^{-m-1} &= \sum_{n=0}^{\infty} \binom{m+n}{n} x^n, \quad |x| < 1. \\
-\ln(1-x) &= \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad -1 \leq x < 1. \\
\arctan(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1. \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x. \\
\cos(x) = \mathrm{Re}(e^{ix}) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \forall x. \\
\sin(x) = \mathrm{Im}(e^{ix}) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \forall x.
\end{aligned}$$

This handout can (soon) be found at

http://math.stanford.edu/∼vakil/putnam03/

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