PUTNAM PROBLEM SOLVING SEMINAR WEEK 4: GAME THEORY (AND SOME LINEAR ALGEBRA)

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Problems in Game Theory.

Explain who can win the following games by playing perfectly, and how. (Test your arguments by playing against other people.) Many involve *matchstick games*: Several piles of matches are given Two players alternate playing. Each "play" involves removing a certain number of matches from one of the piles. The last person to play wins.

1. Each player can remove between one and three matches. The initial pile has 10 matches.

2. Solve the "misère" version of the previous game: the last person to play *loses*.

3. Each player can remove 2^n matches, for any non-negative integer n.

4. There are four piles of matches, with 7, 8, 9, and 10 matches respectively. Each player can remove between one and three matches from one of the piles.

5. There is one pile of matches. When the pile has n matches left, the next player may remove up to $2\sqrt{n+1}-2$ matches. (The game ends when there is one match left.)

6. There are four piles of matches. If there are n matches in the pile, then the next player may remove 2^m matches, where 2^m appears in the binary representation of n.

7. Show that there is an "optimal strategy" for chess.

8. *The fifteen game. (This one is more a "trick" than a problem.)* Nine cards are on the table, numbered one through nine. The two players alternate picking up cards. The first player to have three cards summing to fifteen wins. If all cards are picked up without either player winning, the game is declared a "draw". Show that (i) if both players play perfectly, the game will be drawn, and (ii) if one player "knows what's going on", she can do very well, for example by starting with any of the even cards. (Big hint for both parts

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of the problem: take a 3×3 magic square, and keep track of the picked-up cards there. Notice that you are really playing some other (famous) game.)

9. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking **either**

(a) one bean from a heap, provided at least two beans are left behind in that heap, **or**
(b) a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

10. Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

Problems in Fancy Linear Algebra

11.* *(The Vandermonde determinant)* Prove that

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = (x_1 - x_2) \cdots (x_1 - x_n) \cdots (x_{n-1} - x_n).$$

12.* A k-dimensional parallelepiped in n-dimensional space is generated by the k vectors $\vec{v}_1, \ldots, \vec{v}_k$. Show that its volume is given by

$$\sqrt{\det \begin{pmatrix} \vec{v}_1 \cdot \vec{v}_1 & \cdots & \vec{v}_1 \cdot \vec{v}_k \\ \vdots & \ddots & \vdots \\ \vec{v}_k \cdot \vec{v}_1 & \cdots & \vec{v}_k \cdot \vec{v}_k \end{pmatrix}}.$$

(This formula is handy in many ways. For example, it gives an easy criterion for when k vectors are linearly dependent. Notice that it isn't even a priori clear that the determinant is non-negative!)

13. Calculate

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{pmatrix}$$

14.

(a) Let E_n denote the determinant of the n-by-n matrix having -1's below the main diagonal (from upper left to lower right) and 1's on and above the main diagonal. Show that $E_1 = 1$ and $E_n = 2E_{n-1}$ for $n > 1$.
(b) Let D_n denote the determinant of the n-by-n matrix whose (i, j)th element (the element of the ith row and jth column) is the absolute value of the difference of i and j. Show that $D_n = (-1)^{n-1}(n-1)2^{n-2}$.
(c) Let F_n denote the determinant of the n-by-n matrix with a on the main diagonal, b on the superdiagonal (the diagonal immediately above the main diagonal — having $n-1$ entries), and c on the subdiagonal (the diagonal immediately below the main diagonal — having $n-1$ entries). Show that $F_n = aF_{n-1} - bcF_{n-2}$, $n > 2$. What happens when $a = b = 1$ and $c = -1$?
(d) Evaluate the n-by-n determinant A_n whose (i, j)th entry is $a^{|i-j|}$ by finding a recursive relationship between A_n and A_{n-1}.

15.* A matrix (m_{ij}) is circulant if the entry m_{ij} depends only on $j - i$ modulo n. Find the eigenvectors of a circulant n by n matrix. (Hint: Try the case $n = 2$, and make a guess!)

16. Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}.$$

Is the set $\{D_n/n!\}_{n\geq 2}$ bounded?

17. A sequence of convex polygons $\{P_n\}$, $n \geq 0$, is defined inductively as follows. P_0 is an equilateral triangle with sides of length 1. Once P_n has been determined, its sides are trisected; the vertices of P_{n+1} are the *interior* trisection points of the sides of P_n. Thus P_{n+1} is obtained by cutting corners off P_n, and P_n has $3 \cdot 2^n$ sides. (P_1 is a regular hexagon with sides of length $1/3$.) Express $\lim_{n\to\infty} \mathrm{Area}(P_n)$ in the form $\sqrt{a}/b$, where a and b are positive integers.

18. Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

19. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size such that $\mathbf{ABAB} = \mathbf{0}$, does it follow that $\mathbf{BABA} = \mathbf{0}$?

This handout can (soon) be found at

http://math.stanford.edu/~vakil/putnam03/

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