PROBLEM SOLVING MASTERCLASS WEEK 5

1. A follow-up to Paul's problem from last week (Putnam 1982B6): "Let $A(a,b,c)$ be the area of a triangle with sides a, b, c. Let $f(a,b,c) = \sqrt{A(a,b,c)}$. Prove that for any two triangles with sides a, b, c and a', b', c' we have

$$f(a,b,c) + f(a',b',c') \leq f(a+a', b+b', c+c').$$

When do we have equality?" Show that

$$\begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}$$

has only non-positive eigenvalues. (More generally, how do you find eigenvalues of "circulant" matrices"?) (Ravi)

2. Consider a triangle S in 3-space, and a fixed plane π such that the triangle and the plane do not intersect. Assume the sun is direclty above the plane so that the triangle casts a shadow onto the plane (i.e. the shadow is an orthogonal projection of the triangle onto the plane). Call the image triangle S'. Show that S' always fits inside S. (Kiyoto)

3. Let $P(t)$ be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations

$$0 = \int_0^x P(t)\sin t \, dt = \int_0^x P(t)\cos t \, dt$$

has only finitely many real solutions x. (Alex, Putnam 1980A5)

4. Given an arbitrary triangle, find the circumscribed (inscribed) ellipse with the smallest (largest) area. (Yuanli)

5. Prove that there are unique positive integers a, n such that $a^{n+1} - (a+1)^n = 2001$. (Frank, Putnam 2001A5)

6. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

(Ravi, Putnam 1996B2)

Date: Monday, December 1, 2003.

7. Suppose that a_1, a_2, ... is a sequence of distinct positive integers. Prove that for all positive integers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

(Shrenik, IMO 1978#5)

Follow-up: Let $x_1 \geq x_2 \geq \cdots \geq x_n$, and $y_1 \geq y_2 \geq \cdots \geq y_n$ be real numbers. Prove that if z_i is any permutation of the y_i, then

$$\sum_{i=1}^{n} (x_i - y_i)^2 \leq \sum_{i=1}^{n} (x_i - z_i)^2.$$

(IMO 1975 #1)

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